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Quasi-Classical Dynamics

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Nonlinear Dynamics in Quantum Mechanics
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Quantum←Classical Physical Systems

- We are interested in studying systems, that we call Quantum←Classical, satisfying the following properties:
  - The system is composed of two part, one quantum and the other classical;
  - The quantum part feels the action of the classical one, *e.g.* the latter exerts a force on the former;
  - The action of the quantum part on the classical part is negligible.
- Possible examples are given by:
  - An electron subjected to the electromagnetic field of a nucleus;
  - An atom in an optical lattice, or in another trap;
  - A quantum dot in an electrically driven device;
  - A subatomic particle in a background gravitational field.

Aim

- Derive the Quantum←Classical description as an effective theory, starting from a more fundamental theory.

  In order to do that, we use the following scheme:

  1. We start considering a fully microscopic theory where the to-be-classical part has quantum effects, and feels the action of the to-remain-quantum part.
  2. We introduce a quasi-classical parameter, that measures the "quantumness" of the to-be-classical part only, and we perform the quasi-classical limit in which such quantumness vanishes.
  3. Taking this limit is sufficient, as we will see, to both cancel the quantum effects on the now-classical part, and the action of the remaining-quantum part on the now-classical part.

Mathematical Structure of Quantum←Classical Systems

- Quantum Part:
  - Observables are bounded operators on a Hilbert space $\mathscr{H}$.
  - States are positive trace(=1) class operators on $\mathscr{H}$.
- Classical Part:
  - Phase space $\mathfrak{h}$.
  - Observables are (real- or complex-valued) functions on $\mathfrak{h}$.
  - States are probability measures on $\mathfrak{h}$.

- Action of the Classical Part on the Quantum one:
  - Quantum Observables would in general depend on the configuration $z \in \mathfrak{h}$ of the Classical Part:

$$\mathfrak{h} \ni z \longmapsto \mathscr{F}(z) \in \mathscr{B}(\mathscr{H}) .$$

  - Quantum States also would depend on the configuration of the Classical Part, as well as their evolution:

$$\mathbb{R} \times \mathfrak{h} \ni (t, z) \longmapsto \mathcal{U}_t(z)\,\gamma(z)\,\mathcal{U}_t^*(z) \in \mathscr{L}^1_{+,1}(\mathscr{H})$$

- Action of the Quantum Part on the Classical one:
  - *None*.

Mixed Probabilistic Description:

- For the quantum part it is possible to define the noncommutative probability (state), at time t, $\gamma\big|_{t,E}$, *conditioned to an observed value* λ *of the classical observable* f:

$$\gamma\big|_{t,f=\lambda} = \int_{\mathfrak{h}} \mathbb{1}_{\{f=\lambda\}}(z)\mathcal{U}_t(z)\,\gamma(z)\,\mathcal{U}_t^*(z)\mathrm{d}\mu_{t,\lambda}(z)\;,$$

  where $\mu_{t,\lambda}$ is the disintegration of μ_t w.r.t. f.
- The complete state of the system is described by a *state-valued measure*:

$$\boxed{\mathfrak{m} \in \mathscr{M}\big(\mathfrak{h}\,,\,\mathscr{L}^1_+(\mathscr{H})\big)\;.}$$

  In fact, if $\mathfrak{h}$ is separable, it is possible to decompose any state-valued measure $\mathfrak{m}$ in a scalar (probability) measure $\mu_{\mathfrak{m}}$, and a Radon-Nikodým derivative $\gamma_{\mathfrak{m}}(z) = \frac{\mathrm{d}\mathfrak{m}(z)}{\mathrm{d}\mu_{\mathfrak{m}}(z)} \in \mathscr{L}^1_+(\mathscr{H})$, with $\mathrm{tr}_{\mathscr{H}}\big(\gamma_{\mathfrak{m}}(z)\big) = 1$ for $\mu_{\mathfrak{m}}$-a.a. $z \in \mathfrak{h}$.

The Microscopic Model

- Coupled Quantum System:
  - The Hilbert space consists of two parts, $\mathscr{H}$ for the to-remain-quantum part, $\mathscr{K}_\varepsilon$ for the to-be-classical part.
  - On the to-be-classical part there is a semiclassical structure, that depends on the quasi-classical parameter $\varepsilon \to 0$:

$$[a_\varepsilon(k), a_\varepsilon^*(k')] = \varepsilon\delta(k-k') \; , \; [a_\varepsilon(k), a_\varepsilon(k')] = [a_\varepsilon^*(k), a_\varepsilon^*(k')] = 0 \; .$$

  - The Hamiltonian of the system is of the form

$$\boxed{H_\varepsilon = H_{\mathtt{trq}}(x, -i\nabla) + H_{\mathtt{tbc}}(a_\varepsilon, a_\varepsilon^*) + H_{\mathtt{I}}(x, -i\nabla, a_\varepsilon, a_\varepsilon^*) \; .}$$

  - A state is a coupled density matrix $\Gamma_\varepsilon \in \mathscr{L}_+^1(\mathscr{H} \otimes \mathscr{K}_\varepsilon)$, with quantum evolution

$$\boxed{\Gamma_\varepsilon(t) = e^{itH_\varepsilon}\Gamma_\varepsilon e^{-itH_\varepsilon} \; .}$$

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  - The Hamiltonian of the system is of the form

$$\boxed{H_\varepsilon = H_{\mathtt{trq}}(x, -i\nabla) + \left(\tfrac{1}{\varepsilon}\right) H_{\mathtt{tbc}}(a_\varepsilon, a^*_\varepsilon) + H_{\mathrm{I}}(x, -i\nabla, a_\varepsilon, a^*_\varepsilon)\ .}$$

  - A state is a coupled density matrix $\Gamma_\varepsilon \in \mathscr{L}^1_+(\mathscr{H} \otimes \mathscr{K}_\varepsilon)$, with quantum evolution

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Quasi-Classical Limit of States

Theorem ([Fa 2018, CoFaOl 2019] - Extending [AmNi 2008-])

$$\Gamma_\varepsilon \xrightarrow[\varepsilon \to 0]{} \mathfrak{m}$$

Quasi-Classical Limit of States

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$\exists \delta > 0 \exists C > 0 \ \mathrm{Tr}\big(\Gamma_\varepsilon (\int a_\varepsilon^*(k) a_\varepsilon(k) \mathrm{d}k)^\delta\big) \leq C \implies \exists \varepsilon_n \to 0 \, \exists \mathfrak{m} \in \mathscr{M}\big(\mathfrak{h}, \mathscr{L}_+^1(\mathscr{H})\big)$:

$$\Gamma_{\varepsilon_n} \underset{n\to\infty}{\longrightarrow} \mathfrak{m}$$

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- The above convergence holds as a weak-* convergence on $\mathscr{L}^1(\mathscr{H})$ of Fourier transforms: $\forall \xi \in \mathfrak{h}$,

$$\lim_{n\to\infty} \hat{\Gamma}_{\varepsilon_n}(\xi) = \lim_{n\to\infty} \mathrm{tr}_{\mathscr{K}_{\varepsilon_n}}\Big(\Gamma_\varepsilon e^{i\left(a^*_\varepsilon(\xi)+a_\varepsilon(\xi)\right)}\Big) = \int_{\mathfrak{h}} e^{2i\Re\langle \xi, z\rangle_{\mathfrak{h}}} \gamma_{\mathfrak{m}}(z) \mathrm{d}\mu_{\mathfrak{m}}(z) = \hat{\mathfrak{m}}(\xi) \, .$$

- **Warning:**

$$0 \leq \mu_{\mathfrak{m}}(\mathfrak{h}) \leq 1 \, !$$

Quasi-Classical Dynamics: Egorov-type Theorem

Theorem ([CoFaOl 2019])

$$\Gamma_{\varepsilon_n} \underset{n\to\infty}{\longrightarrow} \mathfrak{m} \iff \forall t \in \mathbb{R}\,,\ \Gamma_{\varepsilon_n}(t) \underset{n\to\infty}{\longrightarrow} \mathfrak{m}_t\,.$$

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- $\mathfrak{m}_t$ is given by the Radon-Nikodým decomposition

$$\mathrm{d}\mathfrak{m}_t = \gamma_{\mathfrak{m}_t}(z)\mathrm{d}\mu_{\mathfrak{m}_t}\ ,$$

  where

$$\gamma_{\mathfrak{m}_t} = \mathcal{U}_{t,0}(z)\gamma_{\mathfrak{m}}(z)\mathcal{U}^*_{t,0}(z)\ ,$$

$$\mu_{\mathfrak{m}_t} = \Phi_{t\,\star}\,\mu_{\mathfrak{m}}\ ,$$

  with $\mathcal{U}_{t,0}(z)$ the unitary evolution on $\mathscr{H}$ generated by

$$\mathcal{H} = H_{\mathrm{trq}}(x, -i\nabla) + H_{\mathrm{I}}(x, -i\nabla, \Phi_t\, z, \overline{\Phi_t\, z})\ ,$$

and Φ_t is

$$\Phi_t = \begin{cases} \texttt{Ham. flow gen. by } H_{\texttt{tbc}}(z,\bar{z}) & \texttt{if } (\frac{1}{\varepsilon})H_{\texttt{tbc}}(a_\varepsilon, a_\varepsilon^*) \\ \texttt{id} & \texttt{if } (\ )H_{\texttt{tbc}}(a_\varepsilon, a_\varepsilon^*) \end{cases} .$$

Remarks

- The loss of mass can be only lost at the initial time: $\forall t \in \mathbb{R}$,

$$\mu_{\mathfrak{m}_t}(\mathfrak{h}) = \mu_{\mathfrak{m}}(\mathfrak{h}) .$$

- This description fits very nicely with the requirements asked for a Quantum←Classical theory.

Outline of the Proof

1. Given $\Gamma_{\varepsilon_n} \to \mathfrak{m}$, extract a subsequence $\Gamma_{\varepsilon_{n_k}}$ such that $\forall t \in \mathbb{R}$, $\Gamma_{\varepsilon_{n_k}}(t) \to \mathfrak{m}_t$.
2. By newly developed techniques of semiclassical analysis, study the limit $k \to \infty$ of the equation:
$$\Gamma_{\varepsilon_{n_k}}(t) = \Gamma_{\varepsilon_{n_k}} - i \int_0^t \left[H_{\varepsilon_{n_k}}, \Gamma_{\varepsilon_{n_k}}(\tau) \right] \mathrm{d}\tau \,.$$
3. Study the properties of the transport equation for the quasi-classical measure $\mathfrak{m}_t$:
$$\mathrm{d}\mathfrak{m}_t(z) = \mathrm{d}\mathfrak{m}(z) - i \int_0^t \left[\mathcal{H}(x, -i\nabla, \Phi_t z, \overline{\Phi_t z}), \gamma_{\mathfrak{m}_\tau}(z) \right] \mathrm{d}\mu_{\mathfrak{m}_\tau} \, \mathrm{d}\tau \,,$$
in particular the uniqueness of solutions that satisfy suitable regularity proeprties (that we know *a priori* to be satisfied by limit measures).

Thank you for the attention.