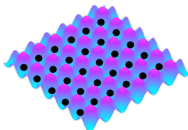


External Potentials Generated by the Interaction with a Semiclassical Field.

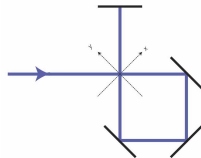
(Joint work with Michele Correggi)

Physical motivation

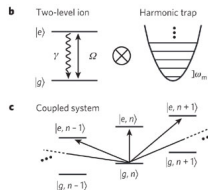
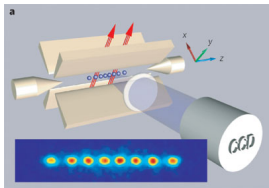
■ Optical lattices



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■ Traps



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∴ Finely tuned external potentials acting on quantum particles
(obtained by the interaction with radiation fields)

¹Picture courtesy of Rainer Blatt and David Wineland. *Nature*, 453(7198) 1008–1015. 2008.

The full quantum system

- $$L^2(\mathbb{R}^{dN}) \otimes \Gamma_s(\mathfrak{H}) \quad , \quad \mathfrak{H} = \begin{cases} \ell^2(\mathbb{Z}^d) & \text{(cavity radiation)} \\ L^2(\mathbb{R}^d) & \text{(free radiation)} \end{cases} .$$

- $$H = \sum_{j=1}^N (-\Delta_j + A(x_j)) + \underbrace{W(x_1, \dots, x_N)}_{L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^+) + P_{\ll 1}(-\Delta)} + \begin{cases} \sum_{n \in \mathbb{Z}^d} |k_n| a_n^* a_n \\ \int_{\mathbb{R}^d} \omega(k) a^*(k) a(k) dk \end{cases} ;$$

$$A(x) = \begin{cases} \sum_{n \in \mathbb{Z}^d} \underbrace{\lambda_n}_{\ell^2(\mathbb{Z}^d)} a_n^* e^{-ik_n \cdot x} + \bar{\lambda}_n a_n e^{ik_n \cdot x} \\ \int_{\mathbb{R}^d} \underbrace{\lambda(k)}_{L^2(\mathbb{R}^d)} a^*(k) e^{-ik \cdot x} + \bar{\lambda}(k) a(k) e^{ik \cdot x} dk \end{cases} .$$

Reduced Hamiltonian

Definition 1 (Partial trace)

X self-adjoint on $L^2 \otimes \Gamma_s$, $q_X(\cdot) = \langle \cdot, X \cdot \rangle$. $\forall \Psi \in \Gamma_s$, we define $\langle X \rangle_\Psi$ to be the operator^a on $L^2(\mathbb{R}^{dN})$ associated to the quadratic form

$$q_{\langle X \rangle_\Psi}(\cdot) = q_X(\cdot \otimes \Psi).$$

^aIn this talk, every partial trace yields a closed and bounded from below quadratic form, hence it defines a unique self-adjoint operator.

Taking the partial trace of the full Hamiltonian we obtain

$$\mathcal{H} := \langle H \rangle_{\Psi} - c = \sum_{j=1}^N (-\Delta_j + V_{\Psi, \lambda}(x_j)) + W(x_1, \dots, x_N)$$

$$c = \langle \Psi, \left\{ \begin{array}{l} \sum_{n \in \mathbb{Z}^d} |k_n| a_n^* a_n \\ \int_{\mathbb{R}^d} \omega(k) a^*(k) a(k) dk \end{array} \right. \Psi \rangle_{\Gamma_S}$$

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Macroscopic Field.

Take a (squeezed) coherent state: $f \in \mathfrak{H}$,

$$\langle C(f), \left\{ \begin{array}{l} \sum_{n \in \mathbb{Z}^d} a_n^* a_n \\ \int_{\mathbb{R}^d} a^*(k) a(k) dk \end{array} \right. C(f) \rangle_{\Gamma_s} = \|f\|_{\mathfrak{H}}^2 .$$

Taking the partial trace of the full Hamiltonian we obtain

$$\mathcal{H} := \langle H \rangle_{\Psi} - c = \sum_{j=1}^N (-\Delta_j + V_{\Psi, \lambda}(x_j)) + W(x_1, \dots, x_N)$$

$$c = \langle \Psi, \left\{ \begin{array}{l} \sum_{n \in \mathbb{Z}^d} |k_n| a_n^* a_n \\ \int_{\mathbb{R}^d} \omega(k) a^*(k) a(k) dk \end{array} \right. \Psi \rangle_{\Gamma_s}$$

Macroscopic Field.

Take a (squeezed) coherent state: $f \in \mathfrak{H}$,

$$\langle C(\frac{1}{\sqrt{\varepsilon}} f), \left\{ \begin{array}{l} \sum_{n \in \mathbb{Z}^d} a_n^* a_n \\ \int_{\mathbb{R}^d} a^*(k) a(k) dk \end{array} \right. C(\frac{1}{\sqrt{\varepsilon}} f) \rangle_{\Gamma_s} = \frac{1}{\varepsilon} \|f\|_{\mathfrak{H}}^2 .$$

Taking the partial trace of the full Hamiltonian we obtain

$$\mathcal{H}_\varepsilon := \langle H_\varepsilon \rangle_{\Psi_\varepsilon} - c_\varepsilon = \sum_{j=1}^N (-\Delta_j + V_{\Psi_\varepsilon, \lambda}(x_j)) + W(x_1, \dots, x_N)$$

$$c_\varepsilon = \langle \Psi_\varepsilon, \left\{ \begin{array}{l} \sum_{n \in \mathbb{Z}^d} |k_n| a_n^*(\varepsilon) a_n(\varepsilon) \\ \int_{\mathbb{R}^d} \omega(k) a_\varepsilon^*(k) a_\varepsilon(k) dk \end{array} \right. \Psi_\varepsilon \rangle_{\Gamma_S}$$

Macroscopic Field.

Take a (squeezed) coherent state: $f \in \mathfrak{H}$,

$$\langle C_\varepsilon(\frac{1}{\varepsilon} f), \left\{ \begin{array}{l} \sum_{n \in \mathbb{Z}^d} a_n^*(\varepsilon) a_n(\varepsilon) \\ \int_{\mathbb{R}^d} a_\varepsilon^*(k) a_\varepsilon(k) dk \end{array} \right. C_\varepsilon(\frac{1}{\varepsilon} f) \rangle_{\Gamma_S} = \|f\|_{\mathfrak{H}}^2.$$

$$[a_\varepsilon(f), a_\varepsilon^*(g)] = \varepsilon \langle f, g \rangle_{\mathfrak{H}}.$$

$\lim_{\varepsilon \rightarrow 0} \mathcal{H}_\varepsilon$: semiclassical analysis of the quantum field

Suppose that²

$$\langle \Psi_\varepsilon, \cdot \Psi_\varepsilon \rangle_{\Gamma_s} \xrightarrow[\varepsilon \rightarrow 0]{\mathfrak{T}} \mu \in \mathcal{M}_{\text{rad}}^2(\mathfrak{H}).$$

In addition, define for a.a. $x \in \mathbb{R}^d$, and any $\alpha \in \mathfrak{H}$:

$$\mathcal{A}(x, \alpha) = \begin{cases} \sum_{n \in \mathbb{Z}^d} \lambda_n \bar{\alpha}_n e^{-ik_n \cdot x} + \bar{\lambda}_n \alpha_n e^{ik_n \cdot x} \\ \int_{\mathbb{R}^d} \lambda(k) \bar{\alpha}(k) e^{-ik \cdot x} + \bar{\lambda}(k) \alpha(k) e^{ik \cdot x} dk \end{cases}.$$

²See e.g. Zied Ammari and Francis Nier. *Ann. Henri Poincaré*, 9(8) 1503-1574. 2008 ;

Theorem 1 (Norm resolvent convergence of \mathcal{H}_ε)

$$\mathcal{H}_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\|\cdot\|_{-res}} \mathcal{H}_0(\mu) \in \text{SelfAdj}(L^2(\mathbb{R}^{dN})) ,$$

$$\mathcal{H}_0(\mu) = \sum_{j=1}^N \left(-\Delta_j + \int_{\mathfrak{H}} \mathcal{A}(x_j, \alpha) d\mu(\alpha) \right) + W(x_1, \dots, x_N)$$

- **Cavity radiation.** *Optical lattices:* $\exists \mu_* \in \mathcal{M}_{\text{rad}}^2(\ell^2)$, $\exists \lambda_* \in \ell^2$,

$$V_{\mu_*, \lambda_*}(x) = \sum_{n \in \mathbb{Z}^d} (b_n \cos(k_n \cdot x) + c_n \sin(k_n \cdot x)) .$$

- **Nelson.** $\forall \mu \in \mathcal{M}_{\text{rad}}^2(L^2)$, $\forall \lambda \in L^2$,

$$V_{\mu, \lambda} \in C_\infty(\mathbb{R}^d) .$$

- **Polaron.** $\forall \mu \in \mathcal{M}_{\text{rad}}^2(L^2)$,

$$V_{\mu, |k|^{-\frac{1-d}{2}}} \in P_{\ll 1}(-\Delta) .$$

No trapping!

Trapping potentials

To obtain trapping, we need more “singular” states of the field:

$$\langle \Psi_\varepsilon, \cdot \Psi_\varepsilon \rangle_{\Gamma_s} \xrightarrow[\varepsilon \rightarrow 0]{\mathfrak{T}} M \in \mathcal{M}_{\text{cyl}}(L^2).$$

Concrete (simple) case.

$$\Psi_\varepsilon = C_\varepsilon \left(\frac{1}{\varepsilon} f_\varepsilon \right) = e^{a_\varepsilon^*(f_\varepsilon) - a_\varepsilon(f_\varepsilon)} \Omega, \quad \mathcal{D}(\mathbb{R}^d) \ni f_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\mathcal{D}'_{\text{uw}}} f_0 \in \mathcal{D}'(\mathbb{R}^d) \setminus L^2(\mathbb{R}^d) :$$

$$\langle \Psi_\varepsilon, \cdot \Psi_\varepsilon \rangle_{\Gamma_s} \xrightarrow[\varepsilon \rightarrow 0]{\mathfrak{T}} \delta_{f_0} \in \mathcal{M}_{\text{rad}}(\mathcal{D}'), \quad \delta_{f_0}(L^2) = 0.$$

Theorem 2 (Trapping potentials)

$\forall V \in L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^+)$, $\forall \lambda \in L^2$ with multiplicative inverse, $\exists (f_{V,\varepsilon})_{\varepsilon \in (0,1)} \subset \mathcal{D}(\mathbb{R}^d)$ (explicit) such that w.r.t. $C(\frac{1}{\varepsilon}f_\varepsilon)$:

$$\mathcal{H}_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{s\text{-res}} \sum_{j=1}^N (-\Delta_j + V(x_j)) + W(x_1, \dots, x_N).$$

In particular, $\sum_{j=1}^N (-\Delta_j + x_j^2) + W(x_1, \dots, x_N)$ can be obtained.

■ Price to pay: $c_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} \infty!$

(For linearly coupled systems, trapping requires a large amount of energy from the radiation field).

Quasi-classical characterization of the ground state energy

- Can we characterize the effective ground state energy

$$\lim_{\varepsilon \rightarrow 0} \underline{\sigma}(H)$$

in terms of the ground state energies of $\mathcal{H}_0(\mu)$ (for suitable μ)?

Theorem 3

Let $\omega(k) = \begin{cases} \sqrt{k^2 + 1} \\ 1 \end{cases}$, $L^2 \supseteq \omega^{1/2}L^2 = \{f \in \mathcal{S}', \omega^{1/2}f \in L^2\}$. Then

$$\lim_{\varepsilon \rightarrow 0} \underline{\sigma}(H) = \inf_{\mu \in \mathcal{M}_{\text{rad}}^2(\omega^{1/2}L^2)} \underline{\sigma}(\mathcal{H}_0(\mu) + c_0(\mu)).$$

$$c_0(\mu) = \int_{\omega^{1/2}L^2} \|\omega^{1/2}\alpha\|_2^2 d\mu(\alpha).$$