BOHR'S CORRESPONDENCE PRINCIPLE IN THE NELSON MODEL

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Notes of the talk. Based on a joint work with Zied Ammari [1].

I. **Bohr's correspondence principle in mathematics.** When we talk about the correspondence principle, we mean the following quantum-classical dictionary.

	Quantum (Non-Commutative)	$\hbar \to 0$	*Classical (Commutative)*
States	Non-Comm. probabilities	$\longrightarrow$	Class. probabilities
Observables	Non-Comm. Random Variables	$\longrightarrow$	Class. Random Variables
Dynamics	Unitary linear evolution	$\longrightarrow$	Nonlinear Hamiltonian flow

Bohr's correspondence principle is necessary for a quantum theory to be in agreement with observation (since at macroscopic scales systems behave commutatively). For Quantum Field Theories however, even at the formal level it is not clear whether the correspondence principle should hold or not, especially when a renormalization procedure is involved.

In these notes we concisely discuss the correspondence principle for the renormalized model introduced by E. Nelson, that describes non-relativistic bosons in interaction with a scalar relativistic bosonic field with Yukawa coupling. We omit references throughout these notes; the interested reader may consult [1].

II. **The classical system.** The classical motion is described by a system of two coupled equations: one is Schrödinger and the other Klein-Gordon, with non-linear Yukawa coupling.

$$\text{(S-KG[Y])} \qquad \begin{cases} i\partial_t u = -\Delta u + Au \\ (\Box + 1)A = -\lvert u\rvert^2 \end{cases} \\ \begin{cases} u(0) = u_0 \\ A(0) = A_0 \;,\; \partial_t A(0) = \dot{A}(0) \end{cases} .$$

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In these notes we set $M_u = \frac{1}{2}$, $m_A = 1$, and no external potential acting on u; but the results hold in a more general situation.

The system (S-KG[Y]) is known to be globally well-posed on suitable Sobolev spaces, e.g. on $H^1(\mathbb{R}^3,\mathbb{C}) \oplus H^1(\mathbb{R}^3,\mathbb{R}) \oplus L^2(\mathbb{R}^3,\mathbb{R})$. However it is convenient to make a change of variables from the real-valued $(A, \partial_t A)$ to the complex valued α given by $A = \sqrt{2}\mathrm{Re}\mathscr{F}^{-1}(\omega^{-1/2}\alpha)$, $\partial_t A = \sqrt{2}\mathrm{Im}\mathscr{F}^{-1}(\omega^{1/2}\alpha)$. Therefore we obtain

$$\text{(S-KG}_\alpha\text{[Y])} \qquad \begin{cases} i\partial_t u = -\Delta u + A(\alpha)u \\ i\partial_t \alpha = \omega\alpha + \frac{1}{\sqrt{2\omega}}\mathscr{F}(|u|^2) \end{cases} \begin{cases} u(0) = u_0 \\ \alpha(0) = \alpha_0 \end{cases} .$$

Proposition II.1. S-KG$_\alpha$[Y] *is globally well-posed on the energy space* $H^1(\mathbb{R}^3,\mathbb{C}) \oplus \mathscr{F}H^{1/2}(\mathbb{R}^3,\mathbb{C})$ *and on* $L^2(\mathbb{R}^3,\mathbb{C}) \oplus L^2(\mathbb{R}^3,\mathbb{C})$.

In addition, S-KG$_\alpha$[Y] can be viewed as an Hamiltonian system, with energy functional

(1)
$$\mathscr{E}(u,\alpha) = \langle u, -\Delta u\rangle_2 + \langle \alpha, \omega\alpha\rangle_2 + \tfrac{1}{\sqrt{2\pi}^3}\int_{\mathbb{R}^6} \tfrac{1}{\sqrt{2\omega}}(\bar{\alpha}e^{-ik\cdot x} + \alpha e^{ik\cdot x})|u|^2 \mathrm{d}x\mathrm{d}k$$

densely defined on $D(\mathscr{E}) \supseteq H^1 \oplus \mathscr{F}H^{1/2}$.

III. **The quantum system.** The quantum dynamics should be characterized by the following *formal* operator on $\mathscr{H} = \Gamma_s\big(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\big)$:

$$H = \underbrace{\int_{\mathbb{R}^3} \psi^*(x)(-\Delta_x)\psi(x)\mathrm{d}x + \int_{\mathbb{R}^3} a^*(k)\omega(k)a(k)\mathrm{d}k}_{H_0} + \tfrac{1}{\sqrt{2\pi}^3}$$
$$\int_{\mathbb{R}^6} \tfrac{1}{\sqrt{2\omega(k)}}\psi^*(x)\big(a^*(k)e^{-ik\cdot x} + a(k)e^{ik\cdot x}\big)\psi(x)\mathrm{d}x\mathrm{d}k\ ;$$

where $\psi^\#$ and $a^\#$ are the $\hbar$-dependent annihilation/creation operators corresponding to the first and second L^2-space respectively. More precisely, we have $[\psi(x),\psi^*(x')] = \hbar\delta(x-x')$; and $[a(k),a^*(k')] = \hbar\delta(k-k')$. H is not defined as an operator because of the a^*-creation term in the interaction, but $\langle\,\cdot\,, H\,\cdot\,\rangle_{\Gamma_s}$ is a densely defined quadratic form.

To rigorously define the dynamics it is possible to perform a self-energy renormalization. We introduce the fibration $\mathscr{H} = \bigoplus_{n=0}^{\infty}\mathscr{H}_n$ with $\mathscr{H}_n = L^2_s(\mathbb{R}^{3n}) \otimes \Gamma_s\big(L^2(\mathbb{R}^3)\big)$, and the self-adjoint operator H_σ, $\sigma \in \mathbb{R}^+$, with regularized interaction. Then we perform a dressing transformation in order to single out the divergent self-energy. Define the dressing "group" $(e^{-\frac{i}{\hbar}\theta T_\sigma(\sigma_0)})_{\theta\in\mathbb{R}}$, with $\sigma_0 \in \mathbb{R}^+$ and

$$T_\sigma(\sigma_0) = \int_{\mathbb{R}^6} \psi^*(x)\big(a^*(k)g_\sigma(\sigma_0,k)e^{-ik\cdot x} + a(k)\bar{g}_\sigma(\sigma_0,k)e^{ik\cdot x}\big)\psi(x)\mathrm{d}x\mathrm{d}k \;,$$

$g_\sigma(\sigma_0,k) = \mathbb{1}_{\{\sigma_0<|\cdot|\leqslant\sigma\}}(k)g_0(k)$ where $g_0 \in L^2(\mathbb{R}^3)$ is suitably chosen.

Proposition III.1. *$0 \leqslant \sigma_0 \leqslant \sigma \leqslant \infty \Rightarrow T_\sigma(\sigma_0)$ self-adjoint.*

Finally, we define the dressed Hamiltonian

$$\hat{H}_\sigma(\sigma_0) = e^{\frac{i}{\hbar}T_\sigma(\sigma_0)}H_\sigma e^{-\frac{i}{\hbar}T_\sigma(\sigma_0)} - \hbar E_\sigma(\sigma_0)\int_{\mathbb{R}^3}\psi^*(x)\psi(x)\mathrm{d}x \;;$$

where $E_\sigma(\sigma_0) \underset{\sigma\to\infty}{\longrightarrow} -\infty$ is the divergent self-energy.

Theorem III.2. *$\forall n \in \mathbb{N}$, $\exists\sigma_0(n,\hbar)$, $\forall\sigma_0 < \sigma \leqslant \infty$:*

- *$\hat{H}_\sigma(\sigma_0)\big|_{\mathscr{H}_n}$ self-adjoint with domain $\hat{D}_{\sigma,n} \subset Q\big(H_0\big|_{\mathscr{H}_n}\big)$;*
- *$\hat{H}_\sigma(\sigma_0)\big|_{\mathscr{H}_n} \underset{\|\cdot\|\text{-}res}{\overset{\sigma\to\infty}{\longrightarrow}} \hat{H}(\sigma_0)\big|_{\mathscr{H}_n}$ self-adjoint; and the corresponding unitary groups converge strongly.*

We want to extend the defintion of $\hat{H}(\sigma_0)\big|_{\mathscr{H}_n}$ to the whole Fock space $\Gamma_s(L^2 \oplus L^2)$; however this can be done in many ways. We choose the following that is most suited for the limit $\hbar \to 0$.

Theorem/Definition III.1 (Renormalized Hamiltonians)**.** *$\forall\sigma_0 \in \mathbb{R}^+$, $\exists\mathfrak{N}(\sigma_0,\hbar)$ such that*

$$\hat{H}(\sigma_0) := \begin{cases} \hat{H}(\sigma_0)\big|_{\mathscr{H}_n} & n \leqslant \mathfrak{N}(\sigma_0,\hbar) \\ 0 & n > \mathfrak{N}(\sigma_0,\hbar) \end{cases} ,$$

$$H_{\mathrm{ren}}(\sigma_0) := e^{-\frac{i}{\hbar}T_\infty(\sigma_0)}\hat{H}(\sigma_0)e^{\frac{i}{\hbar}T_\infty(\sigma_0)} ,$$

are self-adjoint on $\mathscr{H}$. Given $\sigma_0 \in \mathbb{R}^+$ and $\hbar \in \mathbb{R}^+$, we say that the renormalized dynamics is non-trivial in any sector with at most $\mathfrak{N}(\sigma_0,\hbar)$ non-relativistic bosons. The number $\mathfrak{N}(\sigma_0,\hbar)$ can be explicitly computed; in particular it is proportional to σ_0, and inversely proportional to $\hbar$.

IV. **S-KG$_\alpha$[Y] revisited: classical dressing.** S-KG$_\alpha$[Y] is the Hamiltonian equation corresponding to the energy functional $\mathscr{E}$ defined in (1). We denote by $\mathbb{E}(\cdot) : \mathbb{R}\times(H^1\oplus\mathscr{F}H^{1/2}) \to H^1\oplus\mathscr{F}H^{1/2}$ the corresponding Hamiltonian flow in the energy space. In other words, $\mathbb{E}(t)(u_0,\alpha_0)$ is the solution at time t of S-KG$_\alpha$[Y].

Now we introduce a group of nonlinear symplectic transformation on the energy space, called classical dressing. Let the functional $\mathscr{D}_{g_\infty(\sigma_0)}$:

$L^2 \oplus L^2 \to \mathbb{R}$ be defined as follows:

$$\mathscr{D}_{g_\infty(\sigma_0)}(u,\alpha) = \int_{\mathbb{R}^6} \bigl(g_\infty(\sigma_0,k)\bar{\alpha}(k)e^{-ik\cdot x} + \bar{g}_\infty(\sigma_0,k)\alpha(k)e^{ik\cdot x}\bigr)|u(x)|^2 \mathrm{d}x\mathrm{d}k \;.$$

Then the corresponding Hamiltonian flow $\mathbb{D}_{g_\infty(\sigma_0)}(\theta) : H^1 \oplus \mathscr{F}H^{1/2} \circlearrowleft$ for any $\theta \in \mathbb{R}$, and it has an explicit and easy form whenever g has fixed parity.

Remark IV.1. Using the standard Wick quantization, we obtain the following very interesting results:

- $(\mathscr{E})^{\mathrm{Wick}} = \langle\,\cdot\,, H\,\cdot\,\rangle$ (not well-defined);
- $\bigl(\hat{\mathscr{E}}(\sigma_0)\bigr)^{\mathrm{Wick}} := \bigl(\mathscr{E} \circ \mathbb{D}_{g_\infty(\sigma_0)}(-1)\bigr)^{\mathrm{Wick}} = \langle\,\cdot\,, \hat{H}(\sigma_0)\,\cdot\,\rangle$ (renormalized and well-defined on any sector with at most $\mathfrak{N}(\sigma_0,\hbar)$ non-relativistic bosons);
- $\mathbb{E}(t) = \mathbb{D}_{g_\infty(\sigma_0)}(1) \circ \hat{\mathbb{E}}(\sigma_0,t) \circ \mathbb{D}_{g_\infty(\sigma_0)}(-1) \underset{\hbar\to 0?}{\overset{\mathrm{Quant}}{\rightleftarrows}} e^{-\frac{i}{\hbar}tH_{\mathrm{ren}(\sigma_0)}} = e^{-\frac{i}{\hbar}T_\infty(\sigma_0)}e^{-\frac{i}{\hbar}t\hat{H}(\sigma_0)}e^{\frac{i}{\hbar}T_\infty(\sigma_0)}$.

Therefore $\hat{\mathscr{E}}(\sigma_0)$ seems to be the form of the energy most suitable for quantization.

V. **The "classical" meaning of σ_0.** $\inf_{(u,\alpha)\in D(\mathscr{E})} \mathscr{E}(u,\alpha) = -\infty$; on the other hand $\inf_{\substack{(u,\alpha)\in D(\mathscr{E})\\ \|u\|_2\leqslant\sqrt{\mathfrak{C}}}} \mathscr{E}(u,\alpha) > -\infty$.

Since $\mathbb{E}(t)$ preserves the L^2-norm (mass) of Schrödinger's equation, the constraint $\|u\|_2 \leqslant \sqrt{\mathfrak{C}}$ that makes the energy bounded below is a natural assumption. It is also natural to look for quantum configurations that make the classical energy bounded from below, *i.e.* we consider to be admissible families of quantum states only those families whose classical limits are probability measures in $\mathscr{M}(L^2 \oplus L^2)$, concentrated inside the "ball"

$$B_{\mathfrak{C}}(u) = D(\mathscr{E}) \cap \Bigl\{(u,\alpha) \in L^2 \oplus L^2, \|u\|_2 \leqslant \sqrt{\mathfrak{C}}\Bigr\} \;.$$

We remark that this is only a necessary condition, since there may be families of quantum states whose limits are all concentrated inside $B_{\mathfrak{C}}(u)$, but have an unbounded from below or undefined quantum energy.

Proposition V.1. *Any state $\varrho_\hbar$ on $\mathscr{H}$ with at most $[\mathfrak{C}/\hbar] \in \mathbb{N}$ non-relativistic particles can be written as a linear combination:*

$$\varrho_\hbar = \sum_{i\in\mathbb{N}} \lambda_i(\hbar)|\psi_i(\hbar)\rangle\langle\psi_i(\hbar)| \;,$$

where each $\psi_i(\hbar) \in \mathscr{H}$ *has non-zero components only on* $\bigoplus_{n=0}^{[\mathfrak{C}/\hbar]} \mathscr{H}_n$. *In addition, if* $\varrho_\hbar$ *has at most* $[\mathfrak{C}/\hbar] \in \mathbb{N}$ *non-relativistic particles, then*

$$\varrho_{\hbar_k} \underset{k\to\infty}{\overset{\hbar_k\to 0}{\to}} \mu \in \mathscr{M}(L^2 \oplus L^2) \Rightarrow \mu \text{ is concentrated inside } B_{\mathfrak{C}}(u) \ .$$

Finally, $(\varrho_\hbar)_{\hbar\in(0,1)}$ *satisfies*

$$(\mathrm{A}_0) \qquad (\forall k \in \mathbb{N})\ \mathrm{Tr}\Big(\varrho_\hbar\Big(\int_{\mathbb{R}^3} \psi^*(x)\psi(x)\mathrm{d}x\Big)^k\Big) \leqslant \mathfrak{C}^k \ .$$

In the light of the above, it would be suitable to have a way of defining – for any $\mathfrak{C} > 0$ – the quantum dynamics on the relevant sector $\bigoplus_{n=0}^{[\mathfrak{C}/\hbar]} \mathscr{H}_n$. *This is possible, uniformly in* $\hbar$, *exploiting the freedom of choice of* σ_0: it is sufficient to choose a σ_0 satisfying

$$\Big(\big[\tfrac{\sigma_0-2M}{2\hbar} - 1\big] =\Big)\ \mathfrak{N}(\sigma_0,\hbar) \geqslant [\mathfrak{C}/\hbar] \ .$$

Here M is a constant that depends only on the parameters of $\mathscr{E}$ (masses and coupling constant, that are all fixed in these notes). Therefore the choice of σ_0 is in some sense constrained by the physical requirement that the classical energy should be bounded from below.

VI. **Bohr's correspondence principle.** We are now ready to give a precise meaning to the quantum-classical dictionary of Section I. We make the following assumptions on quantum states: the first is assumption (A_0) above, the second is the following

(A_ϱ)

$$\big(\exists K > 0\big)\big(\forall \hbar \in (0,1)\big)\ \mathrm{Tr}\Big(\varrho_\hbar\Big(\int_{\mathbb{R}^3} \psi^*(x)\psi(x)\mathrm{d}x + \int_{\mathbb{R}^3} a^*(k)a(k)\mathrm{d}k + e^{-\frac{i}{\hbar}T_\infty(\sigma_0)} H_0 e^{\frac{i}{\hbar}T_\infty(\sigma_0)}\Big)\Big) \leqslant K \ .$$

The latter assumption means, roughly speaking, that the family of states has uniformly bounded mass and dressed free energy density.

Theorem VI.1 (Ammari - F. 2016)**.** *Let* $\mathfrak{C} > 0$, *and let* $\sigma_0(\mathfrak{C})$ *be such that* $e^{-\frac{i}{\hbar}tH_{\mathrm{ren}}(\sigma_0)}$ *is non-trivial on any state with at most* $[\mathfrak{C}/\hbar]$ *non-relativistic bosons. If* $(\varrho_\hbar)_{\hbar\in(0,1)}$ *is a family of quantum states satisfying* (A_0) *and* (A_ϱ), *then the correspondence principle holds for evolved states:*

$$\varrho_{\hbar_k} \underset{k\to\infty}{\overset{\hbar_k\to 0}{\to}} \mu \ \Leftrightarrow\ \underbrace{e^{-\frac{i}{\hbar_k}tH_{\mathrm{ren}}(\sigma_0)}\varrho_{\hbar_k}e^{\frac{i}{\hbar_k}tH_{\mathrm{ren}}(\sigma_0)}}_{\varrho_{\hbar_k}(t)} \underset{k\to\infty}{\overset{\hbar_k\to 0}{\to}} \mathbb{E}(t)_\#\mu\,,\ \forall t \in \mathbb{R} \ .$$

Corollary VI.2 (Informal)**.** *For suitably regular densely defined classical observables* $b : L^2 \oplus L^2 \supset D(b) \to \mathbb{R}$, *and suitable quantization procedures*

$^{\mathrm{Quant}_\hbar}$, *the correspondence principle holds (weakly) for observables:*

$$\varrho_{\hbar_k} \underset{k\to\infty}{\overset{\hbar_k\to 0}{\to}} \mu \;\Leftrightarrow\; \mathrm{Tr}\Big(\varrho_{\hbar_k}(t) b^{\mathrm{Quant}_\hbar}\Big) \underset{k\to\infty}{\overset{\hbar_k\to 0}{\to}} \int_{D(b)} b(u,\alpha)\mathrm{d}\big(\mathbb{E}(t)_{\#}\mu\big)(u,\alpha), \forall t\in\mathbb{R}\;.$$

Remark VI.3. With the notation $\varrho_{\hbar_k} \to \mu$ it is meant that the generating functional $\mathscr{G}_{\varrho_{\hbar_k}} : L^2 \oplus L^2 \to \mathbb{C}$ of $\varrho_{\hbar_k}$ converges to the Fourier transform $\mathscr{F}\mu : L^2 \oplus L^2 \to \mathbb{C}$ of a unique probability measure μ.

VII. **Outline of the proof.** The idea is to exploit the classical identity $\mathbb{E}(t) = \mathbb{D}_{g_\infty(\sigma_0)}(1) \circ \hat{\mathbb{E}}(\sigma_0, t) \circ \mathbb{D}_{g_\infty(\sigma_0)}(-1)$ to relate the dressed and undressed evolution. This is of crucial importance since we have an explicit form only for $\hat{H}(\sigma_0)$ (as a quadratic form).

The core of the proof is to prove the convergence:

$$\varrho_{\hbar_k} \underset{k\to\infty}{\overset{\hbar_k\to 0}{\to}} \mu \;\Leftrightarrow\; e^{-\frac{i}{\hbar_k} t\hat{H}(\sigma_0)} \varrho_{\hbar_k} e^{\frac{i}{\hbar_k} t\hat{H}(\sigma_0)} \underset{k\to\infty}{\overset{\hbar_k\to 0}{\to}} \hat{\mathbb{E}}(t)_{\#}\mu\,,\ \forall t\in\mathbb{R}\;. \tag{2}$$

The other steps are a simple combination of the following results:

- $\varrho_{\hbar_k} \underset{k\to\infty}{\overset{\hbar_k\to 0}{\to}} \mu \;\Leftrightarrow\; e^{-\frac{i}{\hbar_k}\theta T_\infty(\sigma_0)} \varrho_{\hbar_k} e^{\frac{i}{\hbar_k}\theta T_\infty(\sigma_0)} \underset{k\to\infty}{\overset{\hbar_k\to 0}{\to}} \mathbb{D}_{g_\infty(\sigma_0)}(\theta)_{\#}\mu$, for any $\theta\in\mathbb{R}$ and $\sigma_0\in\mathbb{R}^+$;
- $\varrho_\hbar(t) = e^{-\frac{i}{\hbar}T_\infty(\sigma_0)} e^{-\frac{i}{\hbar}t\hat{H}(\sigma_0)} e^{\frac{i}{\hbar}T_\infty(\sigma_0)} \varrho_\hbar e^{\frac{i}{\hbar}T_\infty(\sigma_0)} e^{\frac{i}{\hbar}t\hat{H}(\sigma_0)} e^{-\frac{i}{\hbar}T_\infty(\sigma_0)}$;
- $\mathbb{E}(t) = \mathbb{D}_{g_\infty(\sigma_0)}(1) \circ \hat{\mathbb{E}}(\sigma_0, t) \circ \mathbb{D}_{g_\infty(\sigma_0)}(-1)$.

The proof of (2) is obtained as follows. With a term-by-term analysis, we identify the classical limit of the interaction picture integral equation:

$$\mathrm{Tr}\Big(\tilde{\varrho}_{\hbar_k}(t) W_{\hbar_k}(\xi)\Big) = \mathrm{Tr}\Big(\tilde{\varrho}_{\hbar_k} W_{\hbar_k}(\xi)\Big) + \frac{i}{\hbar_k}\int_0^t \mathrm{Tr}\Big(\tilde{\varrho}_{\hbar_k}\big[\big(\hat{H}(\sigma_0) - H_0\big), W_{\hbar_k}(\tilde{\xi}_s)\big]\Big)\mathrm{d}s\;.$$

We thus obtain a transport equation for a classical measure $\tilde{\mu}_t$:

$$\partial_t\tilde{\mu}_t + \nabla^{\mathrm{T}}\big(\mathcal{V}(t)\tilde{\mu}_t\big) = 0\;.$$

This equation is solved by $\tilde{\mu}_t = \mathbb{E}_0(-t)_{\#}\hat{\mathbb{E}}(t)_{\#}\mu_0$. In addition, exploiting the regularity properties of μ_0 (inherited by those of $\varrho(0)$) it is also possible to prove that the aforementioned solution is unique, using optimal transportation techniques.

References

[1] Z. Ammari and M. Falconi. Bohr's correspondence principle in quantum field theory and classical renormalization scheme: the nelson model. *ArXiv e-prints*, 02 2016. URL http://arxiv.org/abs/1602.03212.

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