# An introduction to semiclassical analysis in $\infty$ dimensions, and its applications to wave and quantum field theories.

## i. Introduction

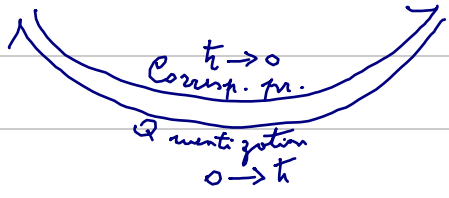
Quantum mechanics is a (noncommutative) probability theory. Classical mechanics is a (classical) probability theory.

* From physics' perspective, QM "$\to$" CM (correspondence principle)

* In mathematical terms, it seems more natural to start with classical objects and deform them to quantum ones (quantization)

Semiclassical analysis aims to study the link between QM and CM in both ways: correspondence principle and quantization.

| | Q (noncomm.) | C (comm.) |
|---|---|---|
| States | Noncommutative probabilities | Classical probabilities |
| Observables | Noncommutative random variables | Classical random variables |
| Dynamics | Unitary linear evolution | Nonlinear Hamiltonian flow |



# ii. Preliminaries (Quantum)

## (A) Algebras of observables.

In quantum mechanics, observables form an algebra. In particular, a $C^*$-algebra.

**Def.** $(\mathcal{A}, +, \cdot, {}^*, \|\cdot\|)$ is a $C^*$ algebra iff:

* $(\mathcal{A}, +)$ is a (complex) linear space;
* $(\mathcal{A}, \|\cdot\|)$ is a Banach space;
* $(\mathcal{A}, +, \cdot)$ is a ($\mathbb{C}$-) algebra;
* ${}^*: \mathcal{A} \to \mathcal{A}$ is an involution, i.e. $\forall a, b \in \mathcal{A}, \forall z \in \mathbb{C}$
  - $(a+b)^* = a^* + b^*$
  - $(ab)^* = b^* a^*$
  - $(a^*)^* = a$
  - $(za)^* = \bar{z}\, a^*$
* $\forall a, b \in \mathcal{A}$:
  - $\|a\| = \|a^*\|$
  - $\|ab\| \leq \|a\| \|b\|$
  - $\|a^* a\| = \|a\|^2$

Why $C^*$-algebras are good candidates to describe collections of quantum observables?

One reason is that $C^*$ algebras are in 1-1 correspondence with algebras of bounded operators on Hilbert spaces:

**Thm.** (Gel'fand-Naimark). Every $C^*$-algebra is isometrically $*$-isomorphic to a self-adjoint algebra of bounded operators on some Hilbert space, closed with respect to the operator norm.

$a = a^*$, $a \geq 0$, $\mathcal{L}(a)$, $\sigma(a)$, $a^{-1}$ can be defined for abstract $C^*$ algebras. The interpretation of these symbols is the same as for bounded operators in Hilbert spaces.

A representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$ consists of a Hilbert space $\mathcal{H}$ and a map $\pi: \mathcal{A} \to \mathcal{L}(\mathcal{H})$ that preserves the $C^*$-algebra structure. $(\mathcal{H}, \pi)$ is irreducible iff $\pi(\mathcal{A})X \subseteq X$ $\Leftrightarrow$ $X = \{0\}$ $\vee$ $X = \mathcal{H}$.

## (B) States.

Let $\mathcal{A}'$ denote the continuous dual of $\mathcal{A}$, endowed with the dual norm $\|\cdot\|' = \sup_{\substack{a \in \mathcal{A} \\ \|a\| = 1}} |\cdot(a)|$.

$\mathcal{A}' \ni \omega \geq 0$ iff $\forall a \in \mathcal{A}$, $\omega(a^* a) \geq 0$.

**Def.** (Quantum states). $\mathcal{S}_{\mathcal{A}} = \{\omega \in \mathcal{A}', \omega \geq 0, \|\omega\|' = 1\}$

## © Weyl $C^*$-algebra of CCR.

**Def.** (Real symplectic linear structure) The couple $(V, \sigma)$ is a real symplectic linear structure iff:

* $V$ is a real linear space

* $\sigma : V \times V \to \mathbb{R}$ ; $\forall v, w \in V$ $\sigma(v,w) = -\sigma(w,v)$; $\forall v, w, x \in V$, $\forall r \in \mathbb{R}$ $\sigma(v+w, rx) = r(\sigma(v,x) + \sigma(w,x))$; $\forall w \in V$ $\sigma(v,w) = 0 \iff v = 0$.

**Rem.** 1) To every complex pre-Hilbert space is associated a real symplectic linear structure:

$$(H, \langle \cdot, \cdot \rangle) \longrightarrow (H_{\mathbb{R}}, \mathrm{Im}\langle \cdot, \cdot \rangle)$$

$\mathbb{C}^d$ (z) $\quad \bar{z}_1 z_2$ $\qquad$ $\mathbb{R}^{2d}$ (Re z, Im z) $\quad \mathrm{Re}\, z_1 \mathrm{Im}\, z_2 - \mathrm{Im}\, z_1 \mathrm{Re}\, z_2$

2) $\dim V < +\infty \iff \exists d \in \mathbb{N}$, $V \cong \mathbb{R}^{2d}$

**Def.** (Weyl $C^*$-algebra) Let $(V, \sigma)$ be a real symplectic linear structure. $\mathcal{W}(V, \sigma)$ is the Weyl $C^*$-algebra associated to $(V, \sigma)$ iff it is the smallest $C^*$-algebra that contains the set

$$\{ W(v),\ v \in V \}$$

satisfying the following properties:

* $\forall v \in V, \quad W(v) \neq 0$

* $\forall v \in V, \quad W(-v) = W(v)^*$

* $\forall v, w \in V, \quad W(v)W(w) = e^{-\frac{i}{2}\sigma(v,w)} W(v+w) = e^{-i\hbar\sigma(v,w)} W(v+w)$

(identity element; ↑ we use the $\hbar$-dependent definition!)

Ex. Prove that: $W(0) = \mathbb{1}$, $W(v)^* = W(v)^{-1}$, $\|W(v)\| = 1$ (unitary elements).

The definition of $\mathcal{W}(V,\sigma)$ is "well-posed" thanks to the following result.

Thm. (Slawny CMP 24, 1972) Fixed $(V,\sigma)$, $\mathcal{W}(V,\sigma)$ is unique up to $*$-isomorphisms.

Ex. (Schrödinger representation). Let $V = \mathbb{R}^{2d}$, $z=(x,\xi)$, $\sigma(z_1,z_2) = x_1\xi_2 - \xi_1 x_2$.

Prove that $(L^2_y(\mathbb{R}^d), \pi_\hbar)$, where

$$\pi_\hbar(W(z)) = e^{-i\sqrt{2}\,\xi\cdot y}\, e^{-i\sqrt{2}\,x\cdot(-i\hbar\nabla_y)}\, e^{i\hbar\xi\cdot x}$$

($-i\hbar\nabla_y$ ↙)

is a representation of $\mathcal{W}(\mathbb{R}^{2d},\sigma)$ (it is also irreducible).

(Fock representation). Let $V = (L^2(\mathbb{R}^d))_{\mathbb{R}}$, $\sigma(\cdot,\cdot) = \mathrm{Im}\langle\cdot,\cdot\rangle_2$.

Prove that $(\Gamma_s(L^2(\mathbb{R}^d)), \pi_f)$, where

$$\pi_{fg}(W(v)) = e^{i(a^*(v)+a(v))}$$

is a (irreducible) representation of $\mathcal{W}(L^2(\mathbb{R}^d)) := \mathcal{W}(L^2_{\mathbb{R}}, \mathrm{Im}\langle\cdot,\cdot\rangle)$.

The Weyl $C^*$-algebra encodes the canonical commutation relations $[q, D_q] = i\hbar$, $[a(v), a^*(w)] = \hbar\langle v, w\rangle$ in exponential form.

**Thm.** (Stone-von Neumann) If $\dim \mathcal{V} < +\infty$, all the irrepresentations are unitarily equivalent (in particular equivalent to the corresponding Schrödinger representation).

**Thm.** (Segal, Trans AMS 88, 1958) If $\dim \mathcal{V} = +\infty$, there is an uncountable number of inequivalent irrepresentations.

## ⓓ Regular states of the Weyl $C^*$-algebra

**Def.** (Regular state) $\omega \in \mathcal{S}_{\mathcal{W}(\mathcal{V},\sigma)}$ is regular iff the $\mathbb{R}$ action on $\omega(W(v))$ is continuous for any $v \in \mathcal{V}$. In other words:

$$\forall v \in \mathcal{V},\ (t \mapsto \omega(W(tv))) \in C^0(\mathbb{R}, \mathbb{C}).$$

**Def.** (Generating functional) $G_\omega(\cdot) := \omega(W(\cdot))$, $\forall \omega \in \mathcal{S}_{\mathcal{W}(\mathcal{V},\sigma)}$.
$(G_\omega : \mathcal{V} \to \mathbb{C})$

**Thm.** (Segal, Canad J. Math. 13, 1961) $G : \mathcal{V} \to \mathbb{C}$ is the generating functional of a (unique) regular state $\omega \in \mathcal{S}_{\mathcal{W}(\mathcal{V},\sigma)}$ iff:

* $\varphi_j(0) = 1$

* $\sum_{j,k \in F} \varphi_j(v_j - v_k) e^{-i\sigma(v_j, v_k)} \bar{\alpha}_k \alpha_j \geq 0 \quad (\alpha_j \in \mathbb{C} \ \forall j \in F, \ F \text{ finite set})$

* The restriction of $\varphi_j$ to any finite dimensional subspace of $V$ is continuous

# iii. Preliminaries (Classical)

## (A) Measures and cylindrical measures on topological vector spaces.

**Def.** Let $E$ be a topological space, $\mathcal{B}(E)$ the corresponding Borel $\sigma$-algebra (generated by open sets).

A finite measure is a map $\lambda : \mathcal{B}(E) \to \mathbb{R}_+$ such that $\lambda(\emptyset) = 0$, $\lambda\left(\bigcup_{n \in \mathbb{N}} b_n\right) = \sum_{n \in \mathbb{N}} \lambda(b_n)$ $(\{b_n\}_{n \in \mathbb{N}} \subset \mathcal{B}(E)$ mutually disjoint).

A (Radon) probability measure is a finite measure $\lambda$ such that $\lambda(E) = 1$ and $\lambda$ is inner regular: $\forall b \in \mathcal{B}(E)$, $\lambda(b) = \sup\{\lambda(K), \ \mathcal{B}(E) \ni K \subseteq b \wedge K \text{ compact}\}$.

Let us now introduce a generalization of measures in infinite dimensional top. vec. spaces: cylindrical measures. Heuristically, cylindrical measures are "almost" measures that become true measures when restricted to finite dimensional subspaces. More rigorously, a cylindrical measure is the

projective system of measures in each finite dimensional subspace.

**Not.** $E$ (real) top. vec. space.

$$\mathcal{F}(E) = \{ V \subset E,\ V \text{ vec. space},\ \dim E/V < +\infty \}.$$

$$p_V : E \to E/V \qquad (x \sim y \text{ iff } x - y \in V)$$
$$x \mapsto [x]$$

If $V, W \in \mathcal{F}(E)$ with $V \supset W$, we denote by $p_{VW} : E/W \to E/V$ the map obtained from the identity passing to the quotients.

**Def.** (Cylindrical measure) Let $E$ be a (real) topological vector space. $\mu$ is a cylindrical measure iff it is a projective system of measures $\mu = (\mu_V, p_{VW})_{V \supset W \in \mathcal{F}(E)}$:

* $\forall V \in \mathcal{F}(E)$, $\mu_V$ is a probability measure on $E/V$

* $\forall V \supset W \in \mathcal{F}(E)$, $\mu_V = p_{VW}(\mu_W)$, i.e.

$$\forall B \in \mathcal{B}(E/V),\quad \mu_V(B) = p_{VW} \# \mu_W(B) = \mu_W(p_{VW}^{-1} B)$$

**Rem.** Every measure is a cylindrical measure. In fact, there is an (injective) map $\sim : \mathcal{P}(E) \to \Psi(E)$:
$$\lambda \mapsto \tilde{\lambda}$$

$$\tilde{\lambda}_V = p_V \# \lambda \qquad \left(\text{together with } p_V = \underset{V \supset W}{p_{VW}} \circ p_W\right)$$

What can we do with cylindrical measures? Not much, apart from integrating cylindrical functions. However, "true" cylindrical measures (belonging to $\mathcal{Y}(E) \setminus \mathcal{P}(E)$) are measures on a bigger space concentrated outside of $E$ (more on this later).

## (B) Fourier transform of a (cylindrical) measure.

Let $E_1, E_2$ be top. vec. spaces; $u \in \mathcal{L}_c(E_1, E_2)$.

$\forall V_2 \in \mathcal{J}(E_2)$, $u^{-1}(V_2) =: V_1 \in \mathcal{J}(E_1)$ ;
(En. prove it)

(passing to quotients) $u_{V_2} : E_1/V_1 \longrightarrow E_2/V_2$ .

$V_2 > W_2 \in \mathcal{J}(E_2)$, $u^{-1}(V_2) =: V_1 \wedge u^{-1}(W_2) =: W_1$
$\Rightarrow V_1 \supset W_2 \wedge$

$$\begin{array}{ccccc} E_1 & \xrightarrow{\pi_{W_1}} & E_1/W_1 & \xrightarrow{\pi_{V_1W_1}} & E_1/V_1 \\ \downarrow u & & \downarrow u_{W_2} & & \downarrow u_{V_2} \\ E_2 & \xrightarrow{\pi_{W_2}} & E_2/W_2 & \xrightarrow{\pi_{V_2W_2}} & E_2/V_2 \end{array}$$

Commutative →

$\therefore \forall \mathcal{Y}(E_1) \ni \mu = (\mu_{V_1})_{V_1 \in \mathcal{J}(E_1)}$

$$\nu_{V_2} := u_{V_2}\left(\mu_{u^{-1}(V_2)}\right) = u_{V_2} \# \mu_{u^{-1}(V_2)}$$

defines $\mathcal{Y}(E_2) \ni \nu = (\nu_{V_2})_{V_2 \in \mathcal{J}(E_2)}$.

$\nu = u(\mu)$ is the image of $\mu$ by $u$. $u' \circ u\,(\mu) = u'(u(\mu))$.

Let $x' \in E'$. $\therefore x' \in \mathcal{L}_c(E, \mathbb{R})$.

Let us denote $\mathcal{P}(\mathbb{R}) \ni \mu_{x'} = x'(\mu) \quad \forall \mu \in \mathcal{P}(E)$.

**Def.** (Fourier transform of $\mu$) $\hat{\mu} : E' \to \mathbb{C}$,

$$\hat{\mu}(x') = \int_{\mathbb{R}} e^{it} \, d\mu_{x'}(t).$$

$\left( \lambda \in \mathcal{P}(E) \Rightarrow \hat{\lambda}(x') = \int_E e^{i x'(x)} \, d\lambda(x). \therefore \hat{\lambda}(x') = \hat{\hat{\lambda}}(x') \right)$

Now $(\mu_{x'})_{\{0\}} = x'_{\{0\}} \# \mu_{x'^{-1}\{0\}}$, where $x'^{-1}\{0\} = \ker x'$.

$\therefore x'_{\{0\}} : E/\ker x' \to \mathbb{R}/\{0\} \cong \mathbb{R}$.

$[x]_{\ker x'} \mapsto [x'(x)]_{\{0\}} \cong x'(x)$

$\Rightarrow \mu_{x'} = x'_{\{0\}} \# \mu_{\ker x'}$.

However, $E' = \bigcup_{V \in \mathcal{F}(E)} V^\circ$, where $V^\circ = \{ x' \in E', \ \forall x \in V \ x'(x) = 0 \}$.

$\therefore \forall x' \in E', \ \exists \tilde{V} \in \mathcal{F}(E), \ x' \in \tilde{V}^\circ \wedge \ker x' \supseteq \tilde{V}$.

$\therefore \mu_{x'} = x'_{\{0\}} \# \mu_{\ker x'/\tilde{V}} \# \mu_{\tilde{V}}$,

$$\int_{\mathbb{R}} e^{it} \, d\mu_{x'}(t) = \int_{E/\tilde{V}} e^{i \, x'_{\{0\}} \circ \pi_{\ker x'/\tilde{V}}(x)} \, d\mu_{\tilde{V}}(x)$$

($\pi_{\ker x'/\tilde{V}}(x) = x'(x)$)

$\therefore \hat{\mu}(x') = \int_{E/\tilde{V}} e^{i x'(x)} \, d\mu_{\tilde{V}}(x)$, $\forall \tilde{V}$ s.t. $x' \in \tilde{V}^\circ$.

Another useful formula is the following.

For any linear map $u : E_1 \to E_2$, denote by ${}^t u : E_2' \to E_1'$ the transposed map defined by:

$${}^t u(\varphi) = \varphi \circ u$$

Then: $\widehat{u(\mu)} = \hat{\mu} \circ {}^t u$ ( $u(\mu)_{\xi'} = \xi'(u(\mu)) = \xi' \circ u(\mu) = {}^t u(\xi')(\mu) = \mu_{{}^t u(\xi')}$ ).

## © Bochner's theorem.

**Def.** (Functions of positive type). Let $G$ be a group (with additive notation). A function $f : G \to \mathbb{C}$ is of positive type iff

$$\sum_{j,k \in F} \alpha_j \bar{\alpha}_k f(x_j - x_k) \geq 0$$

for any finite set $F$, any $\{x_j\}_{j \in F} \subset G$ and $\{\alpha_j\}_{j \in F} \subset \mathbb{C}$.

**Thm.** (Classic Bochner) The Fourier transform is a bijection from $\mathcal{P}(E)$ to the continuous functions of positive type on $E'$ (with $f(0) = 1$), for any finite dimensional vector space $E$.

**proof.** (Reed-Simon II 1975) Since any finite dimensional (real) vector space is canonically isomorphic to some $\mathbb{R}^n$, $n \in \mathbb{N}$, it is sufficient to consider $E = \mathbb{R}^n$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. $\int_{\mathbb{R}^n} \hat{\mu}(\lambda) \varphi(\lambda) \, d\lambda = (2\pi)^{n/2} \int \check{\varphi}(x) \, d\mu(x)$

so the definition of $\hat{\mu}$ coincides with the definition of the (inverse) Fourier transform of a distribution in $\mathcal{S}'(\mathbb{R}^n)$. Therefore the Fourier transform is injective, and $\hat{\mu}$ is a continuous function (dominated convergence). Since $\mu(\mathbb{R}^n) = 1$, $\hat{\mu}(0) = 1$ and $|\hat{\mu}(\lambda)| \leq 1$ for any $\lambda \in \mathbb{R}^n$. Finally, $\sum_{j,k \in F} \alpha_j \bar{\alpha}_k \hat{\mu}(\lambda_j - \lambda_k) = \int_{\mathbb{R}^n} \left| \sum_{j \in F} \alpha_j e^{-i\lambda_j \cdot x} \right|^2 d\mu(x) \geq 0$.

It remains to show that for any continuous $f$ of positive type, $\exists \mu \in \mathcal{P}(\mathbb{R}^n)$ s.t. $\hat{\mu} = f$. Let $\mathcal{K}$ be the set of functions from $\mathbb{R}^n$ to $\mathbb{C}$ that vanish except at a finite set of points.

$$\langle \psi, \varphi \rangle_f = \sum_{x,y \in \mathbb{R}^n} f(x-y)\, \bar{\psi}(y)\, \varphi(x) \qquad , \psi, \varphi \in \mathcal{K}$$

is a "semi"-inner product ($\langle \varphi, \varphi \rangle_f = 0 \not\Rightarrow \varphi = 0$).

$\therefore$ $(\mathcal{K}/\ker\langle \cdot,\cdot \rangle_f\,,\ \langle \cdot,\cdot \rangle_f)$ is a pre-Hilbert space, $\mathcal{H} = \overline{\mathcal{K}/\ker\langle \cdot,\cdot \rangle_f}$.

On $\mathcal{K}$, define $U(t)$, $t \in \mathbb{R}^n$, by $U(t)\varphi(x) = \varphi(x-t)$.
$U(\pm t)$ are isometries on $\mathcal{K}/\ker\langle \cdot,\cdot \rangle_f$, hence extend to a unitary group $(\tilde{U}(t))_{t \in \mathbb{R}^n}$ on $\mathcal{H}$. The group is strongly continuous since $f$ is continuous.

$\therefore$ (Stone) $\langle \tilde{\psi}, \tilde{U}(t)\, \tilde{\varphi} \rangle = \int_{\mathbb{R}^n} e^{i t \cdot \lambda} \langle \tilde{\psi}, dP(\lambda)\, \tilde{\varphi} \rangle_f$.

Now let $\varphi_0(x) = \begin{cases} 1, & x = 0 \\ 0 & \text{otherwise} \end{cases}$.

$\therefore$ $f(t) = \langle [\varphi_0], \tilde{U}(t)[\varphi_0] \rangle_f = \int_{\mathbb{R}^n} e^{i t \cdot \lambda}\, d\mu_f(\lambda)$,

$d\mu_f(\lambda) = \langle [\varphi_0], dP(\lambda)[\varphi_0] \rangle$ +

**Thm.** (Bochner for $\mathcal{F}(E)$) Let $E$ be a real topological vector space. Then the Fourier transform is a bijection from $\mathcal{F}(E)$ to the functions of positive type on $E'$ that are continuous when restricted to any finite dimensional subspace.

**Ex.** Prove the last theorem. (Hint: use the classical Bochner theorem for $\mu_V$, and check the consistency wrt $\mu_{V \subset W}$ by means of $u(\mu) = \hat{\mu} \circ {}^t u$. Injectivity follows from the finite dimensional result using the representation $\hat{\mu}(x) = \int_{E/\tilde{V}} e^{i x'(u)} \, d\mu_V^{\sim}(u)$.)