An introduction to semiclassical analysis in ∞ dimensions, and its applications to mean and quantum field theories.

i. Introduction

Quantum mechanics is a (noncommutative) probability theory. Classical mechanics is a (classical) probability theory.

* From physics' perspective, QM "$\vdash$" CM (correspondence principle)

* In mathematical terms, it seems more natural to start with classical objects and deform them to quantum ones (quantization)

Semiclassical analysis aims to study the link between QM and CM in both ways: correspondence principle and quantization.

	Q (non comm.)	**C (comm.)**
States	Noncommutative probabilities	Classical probabilities
Observables	Noncommutative random variables	Classical random variables
Dynamics	Unitary linear evolution	Nonlinear Hamiltonian flow

$\hbar \to 0$
Corresp. pr.

Quantization
$0 \to \hbar$

ii. Preliminaries (Quantum)

(A) Algebras of observables.

In quantum mechanics, observables form an algebra. In particular, a C^*-algebra.

Def. $(\mathcal{A}, +, \cdot, {}^*, \|\cdot\|)$ is a C^* algebra iff:

* $(\mathcal{A}, +)$ is a (complex) linear space;
* $(\mathcal{A}, \|\cdot\|)$ is a Banach space;
* $(\mathcal{A}, +, \cdot)$ is a ($\mathbb{C}$-) algebra;
* ${}^* : \mathcal{A} \to \mathcal{A}$ is an involution, i.e. $\forall a, b \in \mathcal{A}$, $\forall z \in \mathbb{C}$
  - $(a+b)^* = a^* + b^*$
  - $(ab)^* = b^* a^*$
  - $(a^*)^* = a$
  - $(za)^* = \bar{z} a^*$
* $\forall a, b \in \mathcal{A}$:
  - $\|a\| = \|a^*\|$
  - $\|ab\| \leq \|a\| \|b\|$
  - $\|a^* a\| = \|a\|^2$

Why C^*-algebras are good candidates to describe collections of quantum observables?

One reason is that C^* algebras are in 1-1 correspondence with algebras of bounded operators on Hilbert spaces:

Thm. (Gel'fand-Naimark). Every C^*-algebra is isometrically $*$-isomorphic to a self-adjoint algebra of bounded operators on some Hilbert space, closed with respect to the operator norm.

$a = a^*$, $a \geq 0$, $\rho(a)$, $\sigma(a)$, a^{-1} can be defined for abstract C^* algebras. The interpretation of these symbols is the same as for bounded operators in Hilbert spaces.

A representation $(\mathcal{H}, \pi)$ of $\mathfrak{a}$ consists of a Hilbert space $\mathcal{H}$ and a map $\pi : \mathfrak{a} \to \mathcal{L}(\mathcal{H})$ that preserves the C^*-algebra structure. $(\mathcal{H}, \pi)$ is irreducible iff $\pi(\mathfrak{a}) X \subseteq X$ $\Leftrightarrow$ $X = \{0\} \vee X = \mathcal{H}$.

(B) States.

Let $\mathfrak{a}'$ denote the continuous dual of $\mathfrak{a}$, endowed with the dual norm $\| \cdot \|' = \sup\limits_{\substack{a \in \mathfrak{a} \\ \|a\| = 1}} | \cdot (a) |$.

$\mathfrak{a}' \ni \alpha \geq 0$ iff $\forall a \in \mathfrak{a}$, $\alpha(a^* a) \geq 0$.

Def. (Quantum states). $\mathcal{S}_{\mathfrak{a}} = \{ \omega \in \mathfrak{a}', \omega \geq 0, \|\omega\|' = 1 \}$

c) Weyl C^*-algebra of CCR.

Def. (Real symplectic linear structure) The couple (V,σ) is a real symplectic linear structure iff:

* V is a real linear space

* $\sigma : V\times V \to \mathbb{R}$; $\forall v,w\in V$ $\sigma(v,w) = -\sigma(w,v)$; $\forall v,w,x\in V$, $\forall r\in\mathbb{R}$ $\sigma(v+w, rx) = r(\sigma(v,x) + \sigma(w,x))$; $\forall w\in V$ $\sigma(v,w)=0 \iff v=0$.

Rem. 1) To every complex pre-Hilbert space is associated a real symplectic linear structure:

$$(H, \langle\cdot,\cdot\rangle) \longrightarrow (H_{\mathbb{R}}, \operatorname{Im}\langle\cdot,\cdot\rangle)$$

$\mathbb{C}^d$ $\bar{z}_1 z_2$ $\qquad$ $\mathbb{R}^{2d}$ $\quad$ $\operatorname{Re} z_1 \operatorname{Im} z_2 - \operatorname{Im} z_1 \operatorname{Re} z_2$

(z) $\qquad$ $(\operatorname{Re} z, \operatorname{Im} z)$

2) $\dim V < +\infty \iff \exists d\in\mathbb{N}$, $V \cong \mathbb{R}^{2d}$

Def. (Weyl C^*-algebra) Let (V,σ) be a real symplectic linear structure. $\mathcal{W}(V,\sigma)$ is the Weyl C^*-algebra associated to (V,σ) iff it is the smallest C^*-algebra that contains the set

$$\{ W(v),\ v\in V\}$$

satisfying the following properties:

* $\forall v \in V, \quad W(v) \neq 0$

* $\forall v \in V, \quad W(-v) = W(v)^*$

* $\forall v, w \in V, \quad W(v)\,W(w) = e^{-i\hbar\sigma(v,w)}\,W(v+w)$

¡ we use the ħ-dependent definition !

Ex. Prove that: $W(0)$