iv. Finite dimensional semiclassical analysis

(A) Quantization, Wigner measures. ($\dim V = 2d$, $d \in \mathbb{N}$.)

By Stone - von Neumann's theorem, we can work directly in the Schrödinger representation.

W. l.o.g., let us assume $V = \underset{z=(x,\xi)}{\mathbb{R}^{2d}}$, $\sigma(z_1, z_2) = x_1 \xi_2 - \xi_1 x_2$.

Let $u \in \mathcal{S}(\mathbb{R}^d)$.

$$\begin{aligned} &Op_t(\cdot)\cdot : \mathcal{S}(\mathbb{R}^{2d}) \times \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d) \\ &(t \in [0,1]) \qquad \mathcal{S}'(\mathbb{R}^{2d}) \times \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d) \end{aligned}$$

$$Op_t(a)\, u(y) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} e^{\frac{i}{\hbar}(y-x)\cdot\xi}\, a(ty + (1-t)x, \xi)\, u(x)\, dx\, d\xi.$$

$t = 1$: Kohn - Nirenberg quantization

$\boxed{t = \frac{1}{2} : \text{Weyl quantization}}$

Using such explicit pseudodifferential calculus, many semiclassical results can be proved directly.

References: * Hörmander chap. XVIII, "The Analysis of Linear Partial Diff. Op. III".

* Zworski, Semiclassical analysis 2012

* Martinez, An introduction to semiclassical and microlocal analysis 2002

Let us put Weyl quantization in a form that is more suitable for our purposes:

$$\forall a \in \mathcal{F}^{-1} L^1(\mathbb{R}^{2d}) \subset C_\infty(\mathbb{R}^{2d})$$

$$\mathrm{Op}^{\frac{1}{2}}_\hbar(a) = \int_{\mathbb{R}^{2d}} \underbrace{\hat{a}(x,\xi)}_{\hat{a}(z)} \underbrace{\pi_\Lambda\left(W_\hbar(z=(x,\xi))\right)}_{W_\hbar(z)} \underbrace{\frac{dx\,d\xi}{\nu}}_{d\mathcal{L}(z)}$$

Warning ⚠ The classical proof of the results below goes the other way around (the second is proved, then the first follows). We present them like this to put emphasis on the Wigner measures.

(*) **Thm.** (Wigner measures) Let (V,σ) be a real symplectic linear structure with $\dim V = 2d \in \mathbb{N}$. Let $\omega_\hbar \in \mathcal{S}_{W(V,\sigma)}$ such that $\pi_\Lambda(\omega_\hbar) = \mathrm{Tr}(\rho_\hbar \cdot)$, $\rho_\hbar \in \mathcal{G}^1_+(L^2(\mathbb{R}^d))$ (regular state in the finite dimensional case).

Then $\exists (\hbar_j)_{j\in\mathbb{N}} \subset (0,1)$, $\hbar_j \xrightarrow[j\to\infty]{} 0$ s.t.

$$\mathcal{G}_{\omega_{\hbar_j}} \xrightarrow[j\to\infty]{} \hat{\lambda}, \quad \text{for some } \lambda \in \mathcal{P}(V)$$

(λ identified with its predual)

Thm. (Expectation of Weyl quantizations) Let (V,σ) and $\omega_\hbar$ be as above. Then $\exists (\hbar_j)_{j\in\mathbb{N}} \subset (0,1)$, $\hbar_j \xrightarrow[j\to\infty]{} 0$, and a measure $\lambda \in \mathcal{P}(V)$ s.t.

$$\forall a \in \mathcal{F}^{-1}L^1(V) \quad \omega_{\hbar_j}\left(\mathrm{Op}^{\frac{1}{2}}_{\hbar_j}(a)_{\hbar_j}\right) \xrightarrow[j\to\infty]{} \int_V a(z)\, d\lambda(z).$$

$$\mathbb{E}^{nc}_{\omega_{\hbar_j}}\left(\mathrm{Op}^{\frac{1}{2}}_{\hbar_j}(a)_{\hbar_j}\right) \qquad \mathbb{E}^{c}_{\lambda}(a)$$

proof. Exercise. Hint: $\omega_{\hbar_j}\left(\mathrm{Op}^{\frac{1}{2}}_{\hbar_j}(a)_{\hbar_j}\right) = \int_V \hat{a}(w)\, G_{\omega_{\hbar_j}}(w)\, d\mathcal{L}(w)$

V. Infinite dimensional Wigner measures

(A) Measures as cluster points of nets of regular states.

We want to extend (*) to more general $\mathcal{W}(V,\sigma)$. We are able to prove the following theorem.

Thm (J. 16, arXiv 1605.04778) Let $(\omega_{\hbar_\beta})_{\beta\in B}$ be a net of regular states of $\mathcal{W}(V,\sigma)$. $\hbar_\beta \to 0$

If $\exists V_*$ top. vec. sp. s.t. $(V_*)^* = V$, then:

i. $CCPo\,(\omega_{\hbar_\beta})_{\beta\in B} \neq \emptyset$,

ii. $\mu \in CCPo(\omega_{\hbar_\beta})_{\beta\in B} \Rightarrow \mu \in \mathcal{P}(V_*)$.

} suitable topology

Ideas of the proof:

* $\mathcal{G}\omega_{\pi_\beta}(\cdot)$ resembles a lot the Fourier transform of a cylindrical measure. The only difference is the term $e^{\pm i\sigma(v_j, v_k)}$ in the "positive type" property.

$\mathcal{G}\omega_{\pi_\beta} \xrightarrow[\pi_\beta \to 0]{} \hat{\mu}$

* To make the idea rigorous, up to a subnet extraction, we need an Ascoli result: equicontinuity is needed to preserve continuity in the limit! (when $\mathcal{G}$ is restricted to the fin. dim. subspaces)

* Equicontinuity when restricted to the fin. dim. subspaces is given by thm (*).

* The topology on $\mathrm{Reg}(\mathcal{W}(V,\sigma)) \cup \mathcal{F}(V_*)$ is the one induced by the topology of simple convergence on $\mathbb{C}^V$ by the injections $\mathrm{Reg}(\mathcal{W}(V,\sigma)) \underset{\text{Segal}}{\simeq} \{\text{gen. fun. cont.}\}$, $\mathcal{F}(V_*) \underset{\text{Bochner}}{\simeq} \{\text{func. of pos. type}\}$.

Rem. i) $V_\hbar$, and not V, is the relevant classical space ($\hbar \to 0$?)

(V, σ) carries the structure of a "phase space" (cotangent bundle), however $V_\hbar$ does not. In some sense, $V_\hbar$ is reminiscent of the space of Lagrangian coordinates $X \oplus X$ (as opposed to the phase space $X \oplus X^*$). Hamiltonian and Lagrangian mechanics are not always equivalent for infinite dimensional systems.

ii) There may be more than one classical probability associated to a quantum state $\omega_\hbar$, depending on how we take $\hbar \to 0$ (mathematical problem, not so meaningful physically).

iii) Cylindrical measures, and not just probability measures, are the classical counterpart of (regular) quantum states. ($\hbar \to 0$?)

As we will see, it is indeed possible to obtain some (actually all) "pure" cylindrical measures in $\mathcal{F}(V_\hbar) \setminus \mathcal{P}(V_\hbar)$ as cluster points of nets of regular quantum states.

Cylindrical measures are inconvenient, since it is not possible to integrate functions, unless they're cylindrical as well (i.e. are "supported" on a finite dimensional subspace).

(B) Examples.

All the examples will be discussed for $(V,\sigma) = (L^2(\mathbb{R})_{\mathbb{R}}, \mathrm{Im}\langle\cdot,\cdot\rangle)$.
(Eventually, we will exploit the Fock representation)

a) Squeezed coherent states.

The (Fock) vacuum is the state $\omega_{\Omega_\hbar}$ with generating functional

$$G_{\omega_{\Omega_\hbar}}(f) = e^{-\frac{\hbar}{4}\|f\|_2^2}.$$

The coherent state squeezed around $g \in L^2(\mathbb{R})$ is the state obtained from the vacuum acting with the Weyl operator $W_\hbar\left(\frac{g}{i\hbar}\right)$:

$$C_\hbar(g)(\cdot) = \omega_{\Omega_\hbar}\left(W_\hbar^*\left(\tfrac{ig}{\hbar}\right) \cdot W\left(\tfrac{ig}{\hbar}\right)\right)$$

Ex. Prove that $\pi_g(\omega_{\Omega_\hbar}) = |\Omega\rangle\langle\Omega|$,
$\Omega = (1, 0, \dots, 0, \dots)$ on $\Gamma_s(L^2(\mathbb{R}))$;

Prove that $\pi_g(C_\hbar(g)) = |e^{a^*(g) - a(g)}\Omega\rangle\langle e^{a^*(g) - a(g)}\Omega|$.

Using Weyl's relations:

$$G_{C_\hbar(g)}(f) = e^{i\,\mathrm{Im}\langle f, ig\rangle} e^{-\frac{\hbar}{4}\|f\|_2^2} \xrightarrow[\hbar\to 0]{} e^{i\,\mathrm{Re}\langle f, g\rangle}$$

$$\hat{\delta}_g(f) = \int_{L^2(\mathbb{R})_{\mathbb{R}}} e^{i\langle f, j\rangle}\, d\delta_g(j)$$

$$\therefore \quad C_\hbar(g) \xrightarrow[\hbar \to 0]{} \delta_g \in \mathcal{P}(L^2(\mathbb{R}))$$

b) Multiple cluster points.

Let $g_1, g_2 \in L^2(\mathbb{R})$, $(g_{2,k})_{k \in \mathbb{N}} \subset L^2(\mathbb{R})$, $g_{2,k} \to g_2$.

$$g_k = \begin{cases} g_1 & , k = 2n, n \in \mathbb{N}_* \\ g_{2,k} & , k = 2n+1 \ n \in \mathbb{N}_* \end{cases}.$$

$$B = \mathbb{N}_*, \quad \hbar_k = k^{-1}, \quad k \in \mathbb{N}_*.$$

$$\omega_{\hbar k} = C_{k^{-1}}(g_k).$$

$$\therefore \quad \mathcal{G}_{\omega_{\hbar k}}(f) = e^{i \mathrm{Re} \langle f, g_k \rangle} e^{-\frac{1}{k} \|f\|^2}$$

$$\therefore \quad \mathcal{G}_{\omega_{\hbar 2n}}(f) \xrightarrow[n \to \infty]{} \hat{\delta}_{g_1}(f)$$

$$\mathcal{G}_{\omega_{\hbar 2n+1}}(f) \xrightarrow[n \to \infty]{} \hat{\delta}_{g_2}(f)$$

$$\Rightarrow \quad \mathcal{CC}\, \mathcal{P}_0(\omega_{\hbar k})_{k \in \mathbb{N}} = \{ \delta_{g_1}, \delta_{g_2} \}$$

c) A true cylindrical measure.

Let us consider the free Gibbs state

$$\pi_{\mathcal{G}}(G_\hbar)(\cdot) = \frac{T_{\mathcal{H}}\left(e^{-\frac{\beta \hbar}{\hbar} d\Gamma_\hbar (H_0 - C_\hbar)} \cdot \right)}{T_{\mathcal{H}}\left(e^{-\frac{\beta \hbar}{\hbar} d\Gamma_\hbar (H_0 - C_\hbar)} \right)}$$

where H_0 s-a on $L^2(\mathbb{R})$, s.t. $e^{-\beta_\hbar H_0} \in \mathcal{J}^1(L^2(\mathbb{R}))$ and $\beta_\hbar (H_0 - c_\hbar) > 0$. $\beta_\hbar > 0$ is the ($\hbar$-dep.) inverse temp. and $c_\hbar \in \mathbb{R}$ is the chemical potential.

$$\therefore \mathcal{G}_{G_\hbar}(f) = e^{-\frac{1}{2}\langle f, \frac{\hbar}{2}(1+e^{-\beta_\hbar(H_0-c_\hbar)})(1-e^{-\beta_\hbar(H_0-c_\hbar)})^{-1} f\rangle}$$

[See Bratteli-Robinson vol. 2]

Suppose that $H_0, \beta_\hbar, c_\hbar$ s.t. in addition

$$\frac{\hbar}{2}(1+e^{-\beta_\hbar(H_0-c_\hbar)})(1-e^{-\beta_\hbar(H_0-c_\hbar)})^{-1} \xrightarrow[\hbar\to 0]{} K > 0$$

in the weak operator topology.

$\therefore G_\hbar \to \text{gauss}_K$, where gauss_K is the gaussian cyl. measure on $L^2(\mathbb{R})$ associated to $\langle \cdot, K \cdot \rangle$.

(The gaussian measure is the unique meas. with $\hat{\mu}(\cdot) = e^{-\frac{1}{2}\langle \cdot, K \cdot\rangle}$)

Thm (Minlos) $K \notin \mathcal{J}_+^2(L^2(\mathbb{R}^d)) \Rightarrow \text{gauss}_K \in \mathcal{Y}(L^2) \setminus \mathcal{P}(L^2)$, $\text{gauss}_K \mapsto \tilde{\text{gauss}}_K \in \mathcal{P}(\mathcal{S}')$ s.t. $\tilde{\text{gauss}}_K(L^2) = 0$.

d) $\mathcal{Y}(L^2)$ or $\mathcal{P}(L^2_w)$

Cylindrical measures are inconvenient, for we cannot integrate generic functions, but only cylindrical ones. However $\mathcal{Y}(X)$ can be injected into $\mathcal{P}(\tilde{X})$, for any $X \in \text{Top Vec}_{\mathbb{R}}$ and

a suitable "bigger" $\tilde{X} \supset X$. For $L^2(\mathbb{R}^d)$, the mapping is onto, and $\tilde{L}^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)_w$, the weak completion of L^2. For further details see [Schwartz 73, Rad. mes. ... Tata inst.].

© Surjectivity of the limit.

Thm (T. 16 ortir) $\forall \mu \in \Psi(V_*)$, $\exists (\omega_{n\beta}^{(N)})_{\beta \in B}$ s.t.

$$\omega_{n\beta}^{(N)} \to \mu .$$

This result is easy to prove if $\dim V_* = 2d$, $d \in \mathbb{N}$. The full result is obtained using the one for finite dimensional V_* and the projective structure of cylindrical measures.

Ex. Prove the theorem above if $\dim V_* = 2d$.

Hint: the measures concentrated on a finite set are weakly dense on $\mathcal{P}(V_*)$ $(= \Psi(V_*))$. What happens if we use a linear combination of squeezed coherent states?