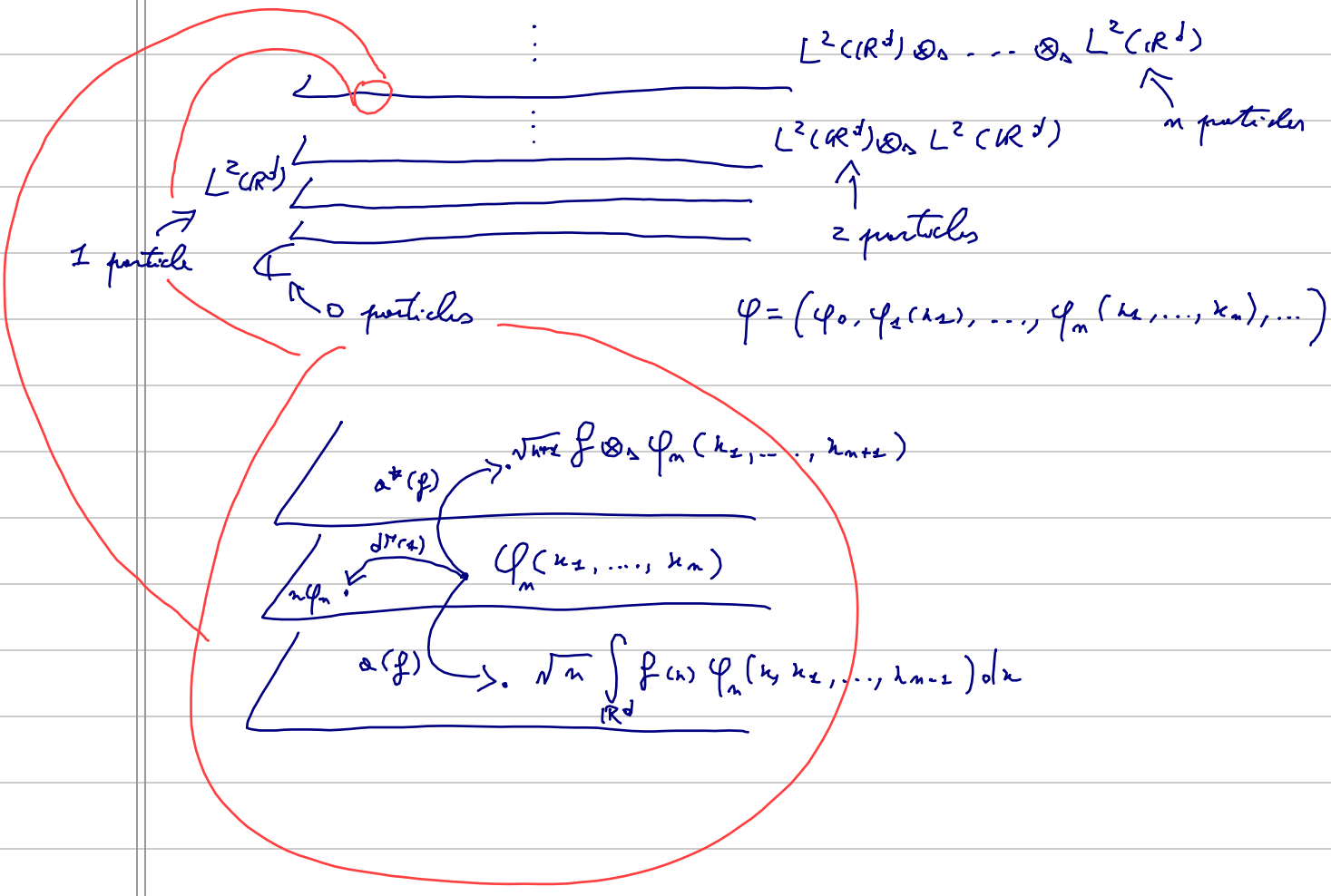
vi. Wick and projective quantizations

For the sake of simplicity, we will only review Wick and projective quantization on $L^2(\mathbb{R}^d)$, but it can be easily generalized to any (separable) Hilbert space $\mathcal{Z}$.

(A) Fock space and Wick quantization of polynomials.

$$\Gamma_s\left(L^2(\mathbb{R}^d)\right) = \bigoplus_{n=0}^{\infty} \underbrace{\left(L^2(\mathbb{R}^d) \otimes_s \dots \otimes_s L^2(\mathbb{R}^d)\right)}_{n \text{ times}} = \bigoplus_{n\in\mathbb{N}} L^2_n$$

$$\underbrace{L^2(\mathbb{R}^d) \otimes_s \dots \otimes_s L^2(\mathbb{R}^d)}_{0 \text{ times}} := \mathbb{C}.$$



Def. $(\mathcal{P}_{p,q}(L^2))$ $P(f) \in \mathcal{P}_{p,q}(L^2)$ iff:

i) $\tilde{P} = \frac{1}{p!}\frac{1}{q!}\partial_f^p \partial_{\bar f}^q P(f) \in \mathcal{L}(L^2_p, L^2_q)$ (Gut/Fu)

ii) $P(f) = \langle \underbrace{f \otimes \dots \otimes f}_{q}, \tilde{P}\, \underbrace{f \otimes \dots \otimes f}_{p} \rangle_{L^2_q}$

Def. $(\mathcal{P}^\infty_{p,q}(L^2))$ $P(f) \in \mathcal{P}^\infty_{p,q}(L^2) \subset \mathcal{P}_{p,q}(L^2)$ iff $\tilde{P} \in \mathcal{T}^\infty(L^2_p, L^2_q)$

Def. (Cylindrical symbols) $B : L^2(\mathbb{R}^d) \to \mathbb{C}$ is a cylindrical symbol iff $\exists P \in \mathbb{P}(L^2)$ and $B' : PL^2 \to \mathbb{C}$ such that

$\forall f \in L^2,\ B(f) = B'(Pf)$.

Here $\mathbb{P}(L^2) = \{P \in \mathcal{L}(L^2),\ P^* = P,\ P^2 = P,\ \dim \operatorname{ran} P < +\infty\}$.

For polynomial symbols, there is a natural Fock-space quantization called Wick quantization:

Def. (Wick quantization) $\forall P \in \mathcal{P}_{p,q}(L^2)$,

$$P^{\mathrm{Wick}}_\hbar : \Gamma_s^{\mathrm{fin}} \to \Gamma_s^{\mathrm{fin}},$$

$$P^{\mathrm{Wick}}_\hbar\Big|_{L^2_n} = \mathbb{1}_{[p,+\infty)}(n)\, \frac{\sqrt{n!(n+q-p)!}}{(n-p)!}\, \hbar^{\frac{p+q}{2}}\, \tilde{P} \otimes_s \mathrm{id}_{L^2_{n-p}}.$$

$$P^{\mathrm{Wick}}_\hbar\Big|_{L^2_n} \in \mathcal{L}(L^2_n, L^2_{n+q-p}).$$

Wick quantization's heuristics (convenient):

$$\big(P(f)\big)^{\text{Wick}}_\hbar = \Big(\big\langle P^{\otimes_q}, \tilde{P}\, f^{\otimes_p}\big\rangle\Big)^{\text{Wick}}_\hbar$$

$$= \left(\int_{\mathbb{R}^{d(q+p)}} \bar{f}(x_1)\cdots\bar{f}(x_q)\, \tilde{P}(x_1,\dots,x_q;y_1,\dots,y_p)\, f(y_1)\cdots f(y_p) \right)^{\text{Wick}}_\hbar$$

$$= \int_{\mathbb{R}^{d(q+p)}} a^*_\hbar(x_1)\cdots a^*_\hbar(x_q)\, \tilde{P}(x_1,\dots,x_q;y_1,\dots,y_p)\, a_\hbar(y_1)\cdots a_\hbar(y_p)$$

($*$: left; $\not{*}$: right)

Examples:

- $d\Gamma_\hbar(1) = \big(\|f\|_2^2\big)^{\text{Wick}}_\hbar$
- $d\Gamma_\hbar(\omega) = \big(\langle f, \omega f\rangle_2\big)^{\text{Wick}}_\hbar$, $\omega^* = \omega \in \mathcal{L}(L^2)$
- $a^\#(g) = \big(\langle g, f\rangle_2^\#\big)^{\text{Wick}}_\hbar$, $g \in L^2$.

(B) Cylindrical Weyl quantization and semiclassics of finite rank polynomial symbols.

Recalling what happens in finite dimensions, it is easy to see how a cylindrical symbol can be Weyl quantized.

Let $B(f) = B'(Pf)$ be cylindrical, with $B' \in F^{-1}L^1(PL^2)$.

$$\mathrm{Op}^{\mathrm{W}}_\hbar\big(B(f)\big) = \int_{PL^2} (\hat{B'})(Pz)\, W_\hbar(z)\, d\mathcal{L}_{PL^2}(Pz).$$

Without entering into the details, the quantization can be extended to any $b \in \mathcal{P}_{p,q}(L^2)$ such that $\tilde{b}$ is of finite rank (i.e. $\exists P \in \mathbb{P}$, $\tilde{b} = P^{\otimes_q} \tilde{b} P^{\otimes_p}$).

Thm. ("Ammari - Nier, Ann. Henri Poincaré 2008")

Let $(\Pi_{\hbar}(\omega_{\hbar}))_{\hbar \in (0,1]} \subseteq \mathcal{D}^1(\Gamma_s(L^2))_+$ be a family of Fock-normal states such that $\mathrm{Tr}\left(\mathcal{N}_{\hbar}(1)^{\frac{p+q}{2}} \Pi_{\hbar}(\omega_{\hbar}) \right) < +\infty$.

Then $\forall b \in \mathcal{P}_{p,q}(L^2)$ such that $\tilde{b}$ is of finite rank,

$$\omega_{\hbar_\beta} \underset{\hbar_\beta \to 0}{\to} \mu \in \mathcal{F}(L^2) \Rightarrow \lim_{\beta \in B} \mathrm{Tr}\left(\mathrm{Op}^{\frac{p}{2}}_{\hbar}(b)_{\hbar_\beta} \Pi_{\hbar}(\omega_{\hbar_\beta}) \right)$$

$$= \int_{L^2/\tilde{V}} \langle P_{\tilde{V}}^{\otimes_p}, \tilde{b}\, P_{\tilde{V}}^{\otimes_q} \rangle \, d\mu_{\tilde{V}}(P_{\tilde{V}}) \,,$$

where $\mathcal{F}(L^2) \ni \tilde{V} = (PL^2)^{\perp}$, seen as a subspace of the dual space of L^2 (that is again L^2).

(c) A priori informations on Wigner measures.

The theorem above lets us gather some a priori informations about the cluster points of Fock-normal states that satisfy some additional regularity assumptions.

Thm (Am Mi 08, 11, 13, Ann. Sc. Nor. Sup. 2015 XIV, J. Math. Pur. Appl. (9) 2011, 55)

a) $\exists \delta > 0$, $\exists C > 0$ s.t. $\mathrm{Tr}\left(d\Gamma_\hbar(A)^{\delta/2} \pi_{\Psi_\hbar}(\omega_\hbar) \right) \leq C$

$\Downarrow$

$\omega_{\hbar_\beta} \xrightarrow[\hbar_\beta \to 0]{} \mu \quad \Rightarrow \quad \mu \in \mathcal{P}(L^2(\mathbb{R}^d))$,

$\int_{L^2} \|u\|_2^\delta \, d\mu(u) \leq C$.

b) $\exists C > 0$ s.t. $\mathrm{Tr}\left(d\Gamma_\hbar(-\Delta + 1) \pi_{\Psi_\hbar}(\omega_\hbar) \right) \leq C$ ($\omega \geq 0$ (s.a.))

$\Downarrow$

$\omega_{\hbar_\beta} \xrightarrow[\hbar_\beta \to 0]{} \mu \quad \Rightarrow \quad \mu \in \mathcal{P}(L^2(\mathbb{R}^d))$, $\mu(H^1(\mathbb{R}^d)) = 1$, ($D(\omega) \cap L^2$)

$\int_{H^1} \|u\|_{H^1}^2 \, d\mu(u) \leq C$.

$\|\omega u\|_2^2 + \|\nabla u\|_2^2$

c) $\exists C > 0$ s.t. $\forall k \in \mathbb{N}$, $\mathrm{Tr}\left[d\Gamma_\hbar(1)^k \pi_{\Psi_\hbar}(\omega_\hbar) \right] \leq C^k$

$\Downarrow$

$\omega_{\hbar_\beta} \xrightarrow[\hbar_\beta \to 0]{} \mu \quad \Rightarrow \quad \mu \in \mathcal{P}(L^2(\mathbb{R}^d))$, $\mu\left(\{ f \in L^2, \|f\| \leq C \} \right) = 1$.

(D) Semiclassics of polynomial with operators with compact symbols.

Thm (Am Mi 08) Let $(\omega_\hbar)_{\hbar \in (0,1]}$ be a family of Fock-Normal states such that the hypothesis a) of the above is satisfied.

Then

$$\omega_{\hbar_\beta} \underset{\hbar_\beta \to 0}{\to} \mu \Rightarrow \forall p,q \in \mathbb{N},\ \forall b \in P^{\infty}_{p,q}(L^2),$$

$$\lim_{\beta \in B} T_{\hbar_\beta}\left(b^{\text{Wick}}_{\hbar_\beta} \pi_\beta(\omega_{\hbar_\beta}) \right) = \int_{L^2} b(z)\, d\mu(z).$$

proof. Lemma. The difference $c^{\text{Wick}}_{\hbar} - Op^{\frac{1}{2}}(c)_{\hbar} = \hbar\, Op^{\frac{1}{2}}(d)_{\hbar}$, for any c polynomial with finite rank and where d is a cylindrical symbol.

The proof is then an exercise (Hint: approximate compact with finite rank, and the procedure is uniform under Wick quantization wrt $\hbar$).

vii. Application to dynamical systems

(A) The Hartree mean field theory.

On $L^2_n(\mathbb{R}^d)$, define

$$H_n = \sum_{j=1}^{n} -\Delta_j + \frac{1}{n} \sum_{i<j} V(x_i - x_j),$$

where $V(x) = V(-x)$, $\overline{V(x)} = V(x)$ $(\forall x \in \mathbb{R}^d)$ and V is infinitesimally $(-\Delta)$-bounded (i.e. $V(x) = \pm \frac{1}{|x|}$)

$H_n = n H|_{L^2_n}$, where H is an operator on $\Gamma_{\frac{1}{n}}(L^2)$ defined by:

$$H = \underbrace{d\Gamma_{\frac{1}{n}}(-\Delta)}_{H_0} + \left(\int_{\mathbb{R}^{2d}} \bar{f}(x) \bar{f}(y) V(x-y) f(x) f(y) \right)^{\text{Wick}}_{\frac{1}{n}}.$$

It is possible to prove that H s.a. $\Rightarrow e^{-\mathrm{i}tH}$ is the unitary evolution that coincides with $e^{-\mathrm{i}tH_n}$ on L^2_n.

Thm (Amour Nier 15). Let $(\varrho_n)_{n\in\mathbb{N}} \subset \mathcal{T}^1_+(L^2_n) \cap \mathcal{D}\left(\sum_{j=1}^{n} -\Delta\right)$. Define $(\omega_n)_{n\in\mathbb{N}}$ as the Fock-normal states on $\left(\Gamma_s(L^2)_{\frac{1}{n}}\right)_{n\in\mathbb{N}}$ s.t.

$$\pi_f(\omega_n)\big|_{L^2_n} = \delta_{m,n}\, \varrho_n .$$

Then

$$\varrho_{n_k} \xrightarrow[k\to\infty]{} \mu \in \mathcal{P}(L^2) \iff \forall t \in \mathbb{R}\ \ e^{-\mathrm{i}n_k t H} \varrho_{n_k} e^{\mathrm{i}n_k t H} \xrightarrow{k\to\infty} \Phi_t \# \mu ,$$

where Φ_t is the Hartree flow on L^2.

(Hartree equation $\mathrm{i}\partial_t u = -\Delta u + (V * |u|^2)u$)

The theorem above is a consequence of

Thm. Let $(\omega_k)_{k\in(0,1]}$ be a family of Fock-normal states on $\Gamma_s(L^2)_k$ such that $\exists C>0$ s.t.

(tt) $$\mathrm{Tr}\left(d\Gamma_k(-\Delta+1)\, \pi_f(\omega_k)\right) \le C .$$

Then

$$\omega_{\hbar_\beta} \xrightarrow[\hbar_\beta \to 0]{} \mu \in \mathcal{P}(L^2) \iff \forall t \in \mathbb{R} \quad e^{-i\frac{t}{\hbar_\beta}H}\, \omega_{\hbar_\beta}\, e^{i\frac{t}{\hbar_\beta}H} \downarrow \hbar_\beta \to 0 \quad \Phi_t \# \mu$$

(B) Step 1 of the proof.

Justify the integral formula:

$$(\star\star) \quad \tilde{\mathcal{G}}_{\omega_\hbar(t)}(\xi) = \mathrm{Tr}\Big(e^{i\frac{t}{\hbar}H_0} e^{-i\frac{t}{\hbar}H} \overbrace{\tilde{\pi}_g(\omega_\hbar)(t)}\, e^{i\frac{t}{\hbar}H} e^{-i\frac{t}{\hbar}H_0}\, \pi_g(W_\hbar(\xi)) \Big)$$

$$\mathrm{Tr}\big(\pi_g(\omega_\hbar)\,\pi_g(W_\hbar(\xi))\big) + \frac{i}{\hbar}\int_0^t \mathrm{Tr}\Big(\tilde{\pi}_g(\omega_\hbar)(s)\,[H-H_0,\ \pi_g(W_\hbar(\Phi_s^0 \xi))] \Big)\, ds$$

where $\Phi_s^0 = e^{is\Delta}$ on L^2.

Use:

$$\star \quad e^{-i\frac{t}{\hbar}H_0}\, \pi_g(W_\hbar(\xi))\, e^{i\frac{t}{\hbar}H_0} = \pi_g(W_\hbar(\Phi_s^0 \xi))$$

$$\star \quad \pi_g(W_\hbar(g))^* \rho_\hbar^{\text{Wick}}\, \pi_g(W_\hbar(g)) = \big(\rho(\cdot + ig)\big)_\hbar^{\text{Wick}}$$

(C) Step 2 of the proof.

Common subsequence extraction:

Under the assumptions, $\tilde{\mathcal{G}}_{\omega_\hbar(t)}(\xi)$ is uniformly equi-

continuous wrt $t \in (0,1]$ on bounded subsets of $\mathbb{R} \times L^2$.

$\therefore$ diag. ext. $\Rightarrow$ $\exists (\omega_{\tau_k})_{k \in \mathbb{N}}$ subseq. s.t.

$\forall t \in \mathbb{R}$ $\quad \tilde{\pi}_t(\omega_{\tau_k})(t) \xrightarrow[k \to \infty]{} \tilde{\mu}_t$

for some $t \mapsto \tilde{\mu}_t$ s.t. $\forall t \in \mathbb{R}$ $\tilde{\mu}_t(H^1) = 1$,
$\mathbb{R} \quad \mathcal{P}(L^2)$

$$\int \|\zeta\|_{H^1}^2 \, d\tilde{\mu}_t(\zeta) \leq C'.$$

this is because $(**) \Rightarrow T_2\left(d\Gamma_{\tau_k}(-\Delta + 1)\, \tilde{\pi}_t(\omega_{\tau_k})\right) \leq C'$.

(D) Limit of the integral formula $(***)$. (Step 3)

Approximate the terms of $\left(\frac{i}{\varepsilon} P_{-}(\cdot + i\varepsilon\zeta_0) - \frac{i}{\varepsilon} P_{-}(\cdot)\right)\Big|_{\varepsilon = 0}$ with compact symbols (and prove the approximation is uniform wrt to once Wick quantized).

(Higher order terms are easily controlled in the limit)

Use the common subsequence to get the limit $\tau_k \to 0$ and obtain the analogues of the weak form of a transport equation for the measure:

$$\partial_t \tilde{\mu}_t + \nabla^T\left(\mathcal{V}(t)\, \tilde{\mu}_t\right) = 0.$$

We know that $\Phi^0_{-t} \# \Phi_t \# \mu_0$ solves the transport equation.

(E) Step 4: prove that the solution is unique.

Optimal transportation techniques of Ambrosio-Gigli-Savaré, 2005, extended by Ammari-Lind 1602.06716.

The a priori informations on $\tilde{\mu}_t$ yielded by the properties of the quantum states prove the following result:

$$\exists (\omega_{\pi_k})_{k\in\mathbb{N}} \text{ subseq. s.t. } \forall t \in \mathbb{R}\ \tilde{\pi}_g(\omega_{\pi_k})(t) \to \tilde{\mu}_t$$

$$\Downarrow$$

$$\tilde{\mu}_t = \Phi^0_{-t} \# \Phi_t \# \mu_0$$

Now suppose that $\omega_{\pi_\beta} \to \mu$. Then $\exists (\omega_{\pi_{\beta_k}})_{k\in\mathbb{N}}$ s.t. $\tilde{\pi}_g(\omega_{\pi_{\beta_k}})(t) \to \tilde{\mu}_t$ $(\mu_0 = \mu)$ $\forall t \in \mathbb{R}$.

In addition, every subsequence $(\omega_{\pi_{\beta_\varepsilon}})_{\varepsilon\in\mathbb{N}}$ that has a subsubsequence such that

$$\forall t \in \mathbb{R}\quad \tilde{\pi}_g(\omega_{\pi_{\beta_{\varepsilon_i}}})(t) \to \nu_t \quad (\nu_0 = \mu)$$

$$\Downarrow$$

$$\nu_t = \tilde{\mu}_t.$$

$\therefore$ Every subseq. has a conv. subsubseq. to the same limit $\Rightarrow \tilde{\pi}_g(\omega_{\pi_\beta})(t) \to \tilde{\mu}_t$.