On the Regularization of Phase-Space Path Integral in Curved Manifolds

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Introduction

In this work we discuss path integrals both in flat and curved space-time: in particular we describe how to write the integral kernel of the time evolution operator (in a curved space time we describe the evolution with respect to a geodesic affine parameter, such as the proper time if the particle has a non-zero mass) in a path integral form, both in configuration and phase space. We briefly introduce a path integral formulation of Quantum Field Theories, however we focus our attention mainly on quantum mechanical models. We develop the idea, originally due to Schwinger, to describe effective actions in quantum field theory as quantum mechanical path integrals of a fictitious particle with evolution dictated by a suitable Hamiltonian function. We will see that one could have obtained the same result as in field theory by first quantizing the particle which actually makes the loop of the Feynman graph corresponding to the effective action.

In the second part we focus on a particle of mass $m \geq 0$ that classically moves along a geodesic of curved space-time. Quantizing this model we will see that even if it is still possible to describe its evolution in the affine parameter by means of a path integral, the results obtained would be ambiguous unless we introduce suitable regularization schemes and related counterterms. This is, in fact, the key point of path integrals used in physics.

Configuration space path integral needs different counterterms, depending on the regularization scheme used, in order to give the same results at any perturbative order for the integral kernel of the evolution operator. These counterterms are finite and univocally fixed by renormalization conditions arising in two-loop calculations. No higher loop calculation are needed since if we consider quantum mechanics in curved space as a field theory, it would be super-renormalizable, which means that possible divergences and ambiguities can appear only up to a finite number of loops.

Different is the case of phase space path integral, as we show in this work: in such functional integral perturbative calculations are not ambiguous nor divergent in the continuum limit, so as far as we limit ourselves to phase space perturbative calculations the introduction of regularization schemes is not particularly useful: they just serve the purpose of explicitly defining the path integral measure, but in many perturbative calculations the regularization can be removed at once. Nevertheless we cannot avoid the regularization procedure completely, for example if we want to integrate out momenta, because in the continuum limit such measure has not a definite meaning. Introducing a cutoff in the Fourier modes of momentum and coordinate paths we calculate explicitly the "counterterms" of phase space path integral in mode regularization.

Contents

Part I

FLAT SPACE-TIME

The aim of this first part is to introduce the reader to the concept of "Path-integral", providing historical remarks as well as its modern derivation; and then to illustrate some of its application, both in Quantum Mechanics and Quantum Field Theory. We will confine ourselves here to flat space-time, for it is easier to develop basic concepts in this environment, and we will see in detail the generalization to an arbitrary curved space-time in the second part of this work.

We start exposing briefly the formulation of R.P. Feynman, who first introduced in 1949 the idea of Path integral inspired by some remarks of P.A.M. Dirac concerning the relation of classical action to quantum mechanics; then we derive Feynman formulas from operator Quantum Mechanics, following a more modern and more general point of view. Technical discussions about functional integration, Trotter formula, Feynman-Kac formula, time-dependent Hamiltonians and Gaussian integrals are developed in the appendices.

outline of first part

After the theoretical derivation of path integrals we develop some applications: we analyze the analytically solvable free and harmonic theories, as well as some perturbative methods for a quite general class of systems.

In the following chapter we develop the mathematical tools necessary to introduce quantum field theory and its formulation through path integrals. We show that the introduction of a particular space, called Q-space, allows us to derive quantum fields path integral in a way perfectly analogous to the one followed in quantum mechanics. Some mathematical technicalities are relegated to the appendices.

Finally, we introduce the effective action for a field interacting with an external background source, and discuss briefly the worldline approach that leads to a description of its one-loop part by means of a quantum mechanical path integral. However we will give a deeper insight of this topic in the second part of this work.

Chapter 1

Path Integrals

"one feels as Cavalieri must have felt calculating the volume of a pyramid before the invention of calculus"

R.P.Feynman

1.1: Some History

As an historical note, we review here Feynman approach to Non-Relativistic Quantum Mechanics, based on Space-Time paths, as he presented it on a famous work[1] in 1948. It provides a "third formulation" of quantum theory, equivalent to both the Schrödinger and Heißenberg one, and inspired by a 1933 Dirac work[2].

1.1.1: Probability Amplitudes

We start presenting its first postulate:

Postulate 1. *If an ideal measurement is performed to determine whether a particle has a path lying in a region R of space-time, then the probability that the result will be affirmative is the absolute square of a sum of complex distributions, the* probability amplitudes, *one for each path in the region:*

Feynman's first postulate

$$\mathscr{P}(R) = |\varphi(R)|^2 \quad , \qquad \text{(probability for region } R\text{)}$$

with

$$\varphi(R) = \lim_{\epsilon \to 0} \int_R \phi(\dots, x_i\,,\, x_{i+1}, \dots) \cdots \mathrm{d}x_i\, \mathrm{d}x_{i+1} \cdots \quad , \qquad \text{(probability amplitude for region } R\text{)}$$

and

$$x_{i+1} \;\; \textit{particle's position at time} \;\; t_{i+1} = t_i + \epsilon \quad .$$

This postulate prescribes the type of mathematical framework required by Heißenberg uncertainty principle for the calculation of probabilities. Now we can imagine that, as ϵ approaches zero, the probability amplitude $\phi(\dots, x_i\,, x_{i+1}, \dots)$ essentially depends on the entire path $x(t)$ rather than on the discrete points x_i of the path. We might call $\Phi[x(t)]$ the *probability amplitude functional* of paths $x(t)$. We can now introduce the second postulate, that gives a prescription to compute this functional for each path:

Postulate 2. *The paths contribute equally in magnitude, but the phase of their contribution is the classical action (in units of $\hbar$);* i.e., *the time integral of the Lagrangian taken along the path:*

Feynman second postulate

$$\Phi[x(t)] \propto e^{\frac{i}{\hbar}S[x(t)]} \quad ,$$

where

$$S[x(t)] = \int \mathscr{L}(\dot{x}(t), x(t))\mathrm{d}t \quad ,$$

and the proportionality is intended up to a normalization global factor.

The Lagrangian is a function of position and velocity, and if we suppose it to be a quadratic function of the velocities we can show the equivalence of these postulates and usual quantum mechanics formulations.

But we have first to interpret this postulate for a discontinuous path, *i.e.* to calculate $\varphi(R)$: we shall assume that $x(t)$ in the interval between t_i and t_{i+1} is the classical path given by Lagrangian $\mathscr{L}$, which starting from x_i at t_i, reaches x_{i+1} at t_{i+1}, at least in the limit $t_i \xrightarrow{\epsilon\to 0} t_{i+1}$. Lagrangian function must therefore depend on no higher time derivatives of the position than the first, in order to be sufficient the specification of start-point and end-point to determine classical path. So for a discontinuous path we can write $S = \sum_i S(x_{i+1}, x_i)$, where

$$S(x_{i+1}, x_i) = \min_{\textit{paths}} \int_{t_i}^{t_{i+1}} \mathscr{L}(\dot{x}(t), x(t))\mathrm{d}t \quad , \tag{1.1.1}$$

and the minimum is taken to evaluate action on classical trajectory. So now we can combine the two postulates and find:

$$\varphi(R) = \lim_{\epsilon\to 0} \int_R e^{\frac{i}{\hbar}\sum_i S(x_{i+1}\,, x_i)} \dots \frac{\mathrm{d}x_{i+1}}{A}\frac{\mathrm{d}x_i}{A} \dots \quad , \tag{1.1.2}$$

where we have split the normalization factor into $1/A$ factors, one for each time slice.

1.1.2: *Wave Function*

In order to check equivalence we have to define the *wave function*, and show that its time development is given by Schrödinger equation.

The wave function is a quantity having a well defined (by Schrödinger equation) time development, so we have to define it by means of successive sections of the path, with respect to time. In order to do that we choose a particular time, say t, and split region R of (1.1.2) into pieces, future and past relative to t. The split is done in this way:

a) a region R', restricted arbitrarily in space, but lying entirely earlier than some time t', such that $t' < t$;

b) a region R'' defined in the same way as in a), but lying later than time t'', such that $t'' > t$;

c) the region between t' and t'', in which all values of x coordinates are unrestricted, *i.e.* all of space-time between t' and t''.

Region c) is not necessary at all, for it can be taken as narrow in time as desired. Nevertheless it is convenient in letting us vary t a little without having to redefine a) and b) .
So we can define the probability $|\varphi(R', R'')|^2$ that if a path *had been* in region R', then *will be* found in region R''. This is the crucial measured quantity in most experiments: we prepare a system in a certain way (*e.g.*, it was in region R') and measure some other property (*e.g.*, will it be found in region R''?).

Now we can use a (1.1.2)-like equation to compute the quantity $\varphi(R', R'')$. Assuming the time t to be one particular point k of the subdivision into ϵ time steps, *i.e.* $t = t_k$, we can split the action exponential in the product of two exponentials:

$$e^{\frac{i}{\hbar}\sum_i S(x_{i+1}, x_i)} \equiv e^{\frac{i}{\hbar}\sum_{i=k}^{\infty} S(x_{i+1}, x_i)} \cdot e^{\frac{i}{\hbar}\sum_{i=-\infty}^{k-1} S(x_{i+1}, x_i)} \quad .$$

This operation is possible essentially because the Lagrangian is a function only of positions and velocities, and so we could split the action into a sum of discretized parts. Now, calling $x_k \equiv x$ and $t_k \equiv t$ we obtain:

$$\varphi(R', R'') = \int \chi^*(x,t)\psi(x,t)\mathrm{d}x \quad , \tag{1.1.3}$$

where

Probability amplitudes and wave functions

$$\psi(x,t) = \lim_{\epsilon\to 0} \int_{R'} e^{\frac{i}{\hbar}\sum_{i=-\infty}^{k-1} S(x_{i+1}, x_i)} \frac{\mathrm{d}x_{k-1}}{A}\frac{\mathrm{d}x_{k-2}}{A}\cdots \quad , \tag{1.1.4}$$

and

$$\chi^*(x,t) = \lim_{\epsilon\to 0} \int_{R''} e^{\frac{i}{\hbar}\sum_{i=k}^{\infty} S(x_{i+1}, x_i)} \frac{1}{A}\frac{\mathrm{d}x_{k+1}}{A}\frac{\mathrm{d}x_{k+2}}{A}\cdots \quad . \tag{1.1.5}$$

The quantity ψ depends only on past information on the system, with respect to t, and it is completely defined when the region R' is known. The information on the future lies entirely on the quantity χ. We have thus defined the concept of wave functions $\psi(x,t)$, $\chi(x,t)$ defining the state of a system.

The quantity $|\int \chi^*(x,t)\psi(x,t)\mathrm{d}x|^2$ is the probability that a system in state ψ will be found by an experiment whose characteristic state is χ, in agreement with ordinary quantum mechanics.

1.1.3: *Equivalence of Formulations*

If the wave functions just defined satisfy Schrödinger equation we have completed our task to show the equivalence between Feynman formulations and the other ones. In order to do that we have to ask that $\mathscr{L}$ is a quadratic, but perhaps inhomogeneous, form in the velocities $\dot{x}(t)$.

If we compute ψ in (1.1.4) at the next time slice we obtain:

$$\psi(x_{k+1}, t+\epsilon) = \int_{R'} e^{\frac{i}{\hbar}\sum_{i=-\infty}^{k} S(x_{i+1},x_i)} \frac{\mathrm{d}x_k}{A}\frac{\mathrm{d}x_{k-1}}{A}\cdots = \int e^{\frac{i}{\hbar}S(x_{k+1},x_k)}\psi(x_k,t)\frac{\mathrm{d}x_k}{A} \quad . \tag{1.1.6}$$

This equation is not exact, it is only true in the limit $\epsilon \to 0$ and we shall derive Schrödinger equation by assuming it is valid to first order in ϵ. In order to do this we need first of all to give an approximate value of (1.1.1), provided the error of the approximation be of an order smaller than the first in ϵ. The Lagrangian is a quadratic form in the velocities, and we will see that the paths important in the calculation are those for which $x_{i+1} - x_i \sim \epsilon^{\frac{1}{2}}$. Under these circumstances, it is sufficient to calculate the integral in (1.1.1) over the classical path taken by a *free* particle. If we chose the coordinates of the system to be *Cartesian*[1,2], the path of a free particle is a straight line. So in this case it is sufficiently accurate to replace the integral by[3]:

$$S(x_{i+1}, x_i) = \epsilon\mathscr{L}\left(\frac{x_{i+1}-x_i}{\epsilon}, x_{i+1}\right) \quad . \tag{1.1.7}$$

[1] more generally, coordinates for which the terms quadratic in the velocities in the Lagrangian function appear with constant coefficients.

[2] if one seeks the differential equation in a different coordinate system, the easiest way is to find the equation in Cartesian coordinates and *then* to transform the coordinate system to the one desired.

[3] this approximation is not valid if we have a vector potential or other terms linear in the velocity in the Lagrangian, if those are the cases we need to use an approximation of the type

$$S(x_{i+1}, x_i) = \frac{\epsilon}{2}\mathscr{L}\left(\frac{x_{i+1}-x_i}{\epsilon}, x_{i+1}\right) + \frac{\epsilon}{2}\mathscr{L}\left(\frac{x_{i+1}-x_i}{\epsilon}, x_i\right)$$

or

$$S(x_{i+1}, x_i) = \epsilon\mathscr{L}\left(\frac{x_{i+1}-x_i}{\epsilon}, \frac{x_{i+1}+x_i}{2}\right) \quad .$$

Thus, in the simple unidimensional case of a particle of mass m moving under a potential $V(x)$, we have:

$$S(x_{i+1}, x_i) = \frac{m\epsilon}{2}\left(\frac{x_{i+1}-x_i}{\epsilon}\right)^2 - \epsilon V(x_{i+1}) \quad .$$

So if we use this last equation and we call $x_{k+1} \equiv x$, $(x_{k+1} - x_k) \equiv \xi$ so that $x_k = x - \xi$, then (1.1.6) becomes:

$$\psi(x, t+\epsilon) = e^{-\frac{i}{\hbar}\epsilon V(x)}\frac{1}{A}\int e^{\frac{im}{2\epsilon\hbar}\xi^2}\psi(x-\xi, t)\mathrm{d}\xi \quad , \tag{1.1.8}$$

so if we Taylor expand around ξ the wave function, perform Gaussian integrations, and consider terms up to first order in ϵ[1], we obtain:

$$\psi(x,t) + \epsilon\partial_t\psi = \left(\frac{2\pi\hbar\epsilon i}{m}\right)^{\frac{1}{2}}\frac{1}{A}\left(1 - \frac{i\epsilon}{\hbar}V(x)\right)\left(\psi(x,y) + \frac{\hbar\epsilon i}{2m}\frac{\partial^2}{\partial x^2}\psi\right) \quad .$$

In order that both sides may agree both to *zero* and to *first* order in ϵ, we must have:

$$A = \left(\frac{2\pi\hbar\epsilon i}{m}\right)^{\frac{1}{2}}$$

and of course

$$i\hbar\partial_t\psi = \frac{1}{2m}(-i\hbar\partial_x)^2\psi + V(x)\psi \quad ,$$

which is the Schrödinger equation we sought. The generalization to arbitrary degrees of freedom is straightforward. For what it concerns χ^*, a differential equation can be developed in the same way, but it appears to have the *time reversed*. However taking the complex conjugates we can see that χ satisfies the same equation as ψ. We have finally shown that Feynman formulation is equivalent[2] to the usual formulation of quantum mechanics provided by Schrödinger or Heißenberg.

We shall conclude here our historic note, and begin to deal with modern formulations of path integral.

1.2: *A modern formulation*

Path integral emerges naturally in usual non-relativistic quantum mechanics as one deals with the dynamics, *i.e.* as one considers evolution of a state through time. In order to know such an evolution in wave functions space, one has to seek the kernel of the following integral equation:

$$\psi(x_f, t_f) = \int \mathrm{d}x_i\, \mathcal{K}(x_f, t_f; x_i, t_i)\, \psi(x_i, t_i) \quad , \tag{1.2.1}$$

[1] Taylor expanding the first member of (1.1.8) and the exponential of the potential up to first order in ϵ.

[2] at least if the system is described by a Lagrangian that is function only of time, position, and velocities and is a quadratic form of the latter (perhaps inhomogeneous).

where x is intended to be a vector of the system's configuration space, and $\mathrm{d}x$ is the infinitesimal volume element of the same space.

If the Hamiltonian is time independent is straightforward to show that $\mathcal{K}$ is a solution of the Schrödinger equation, *i.e.* $\big(i\hbar\partial_t - H(x, -i\hbar\nabla)\big)\mathcal{K}(x,t;x',t') = 0$[1]. Let's see how this kernel can be calculated when we deal with a generic vector of the Hilbert space[2]: suppose that we have a state $|\psi(t)\rangle \in \mathscr{H}$, its time evolution is governed by the evolution operator $\mathcal{U}(t,t')$ in such a way that $|\psi(t_f)\rangle = \mathcal{U}(t_f,t_i)|\psi(t_i)\rangle$. We know that position eigenvectors form a complete set of commuting observables, so we can use the relation $\mathbb{1} = \int \mathrm{d}x_i\, |x_i\rangle\langle x_i|$ to obtain $|\psi(t_f)\rangle = \int \mathrm{d}x_i\, \mathcal{U}(t_f,t_i)|x_i\rangle\langle x_i|\psi(t_i)\rangle$; now finally projecting each member on $|x_f\rangle$ and remembering the usual definition $\langle x|\psi(t)\rangle \equiv \psi(x,t)$ we obtain $\psi(x_f,t_f) = \int \mathrm{d}x_i\, \langle x_f|\mathcal{U}(t_f,t_i)|x_i\rangle\psi(x_i,t_i)$, so from (1.2.1) we immediately identify:

$$\mathcal{K}(x_f,t_f;x_i,t_i) = \langle x_f|\mathcal{U}(t_f,t_i)|x_i\rangle \quad . \tag{1.2.2}$$

Now $\mathcal{U}$ is a unitary operator (in order to conserve probabilities), and can be written in the infinitesimal form $\mathcal{U}(t+\epsilon,t) = \mathbb{1} - i\epsilon\mathcal{H}(t)$, provided $\mathcal{H}$, called the *infinitesimal generator* of the transformation, to be a self-adjoint operator. In fact we can identify $\mathcal{H}(t)$ as the *total Hamiltonian function* of the system. We can also obtain a finite transformation iterating infinitesimal ones, the result being the exponentiation of the generator, provided to be time-ordered to preserve the causality condition[3]:

Hamiltonian-induced time evolution

$$\mathcal{U}(t_f,t_i) = \mathcal{T}\left\{\exp\left[-i\int_{t_i}^{t_f}\mathrm{d}t\,\mathcal{H}(t)\right]\right\} \quad \text{where } \mathcal{T} \text{ stands for time-ordered product} \quad ; \tag{1.2.3}$$

such a relation is known as Dyson series, and has to be intended as an integral series:

$$\mathcal{U}(t_f,t_i) = \mathbb{1} + \sum_{n=1}^{\infty}(-i)^n \int_{t_i}^{t_f}\int_{t_i}^{t_1}\cdots\int_{t_i}^{t_{n-1}} \mathrm{d}t_1\cdots\mathrm{d}t_n\,\mathcal{H}(t_1)\cdots\mathcal{H}(t_n) \quad . \tag{1.2.4}$$

If, as it happens in most cases, the Hamiltonian function does not depend explicitly on time, equation (1.2.3) takes the well known form $\mathcal{U}(t_f,t_i) \equiv \mathcal{U}(t_f-t_i) = \exp\Big[-i(t_f-t_i)\mathcal{H}\Big]$[4], as the time ordering of operators is no longer necessary and the integration is straightforward[5].

Now that the background is set, we can see how to put the propagator $\mathcal{K}(x_f,t_f;x_i,t_i)$ in a path integral form. We will start analyzing the case of a time independent Hamiltonian $\mathcal{H}$, for it happens to be the case in a very wide range of physical situations, in particular those we will treat in greater detail in the rest of the work; nevertheless afterwards we will show how to obtain a path integral propagator even in the case of a time dependent Hamiltonian, although not in the very detail.

[1] for a prove see [3]; for an analogous prove in relativistic quantum mechanics, *i.e.* for Klein-Gordon and Dirac equations, see [4].

[2] from now on we will set $\hbar = 1$.

[3] See [5, 6] for ulterior details.

[4] the two-parameter family of unitary operators $\mathcal{U}(s,t)$ reduces in that case to the *unitary one-parameter group* $\mathcal{U}(t)$, usual representation of time-translations group on a Hilbert space $\mathscr{H}$.

[5] $\mathcal{H}$ does not bear any time dependence and could be carried out.

1.2.1: Time-independent Hamiltonian

In that case relation (1.2.2) becomes:

$$\mathcal{K}(x_f,t_f;x_i,t_i)=\langle x_f|e^{-i(t_f-t_i)\mathcal{H}}|x_i\rangle \quad . \tag{1.2.5}$$

For the sake of simplicity we will consider an usual Hamiltonian $\mathcal{H}(x,p)=\mathscr{T}_0(p)+\mathcal{V}(x)$, where $\mathscr{T}_0$ is a quadratic function of only the momenta and $\mathcal{V}$ is an arbitrary function of the coordinates[1].

Now we can use a formula introduced by H.F.Trotter [7][2]:

$$e^{A+B}=\lim_{n\to\infty}(e^{\frac{1}{n}A}e^{\frac{1}{n}B})^n \tag{1.2.6}$$

Trotter Product Formula

that in our case becomes

$$e^{-i(t_f-t_i)[\mathscr{T}_0(p)+\mathcal{V}(x)]}=\lim_{\epsilon\to0}\underbrace{(e^{-i\epsilon\mathscr{T}_0(p)}e^{-i\epsilon\mathcal{V}(x)})\cdots(e^{-i\epsilon\mathscr{T}_0(p)}e^{-i\epsilon\mathcal{V}(x)})}_{n-1\text{ times}} \quad , \tag{1.2.7}$$

with $\epsilon=(t_f-t_i)/(n-1)$. Now using the two decompositions of identity operator $\mathbb{1}_x=\int \mathrm{d}x\,|x\rangle\langle x|$ and $\mathbb{1}_p=\int\frac{\mathrm{d}p}{(2\pi)^D}|p\rangle\langle p|$ (the subscript $_x$ or $_p$ has no meaning than to distinguish between different decompositions), and defining $x_n\equiv x_f$, $x_0\equiv x_i$, we obtain:

$$\mathcal{K}(x_f,t_f;x_i,t_i)=\lim_{\epsilon\to0}\langle x_n|\underset{\mathbb{1}_{p_n}}{\underset{\downarrow}{e^{-i\epsilon\mathscr{T}_0(p)}}}e^{-i\epsilon\mathcal{V}(x)}\cdots\underset{\mathbb{1}_{x_{n-1}}\cdot\mathbb{1}_{p_{n-1}}}{\underset{\downarrow}{e^{-i\epsilon\mathscr{T}_0(p)}}}e^{-i\epsilon\mathcal{V}(x)}\cdots\underset{\mathbb{1}_{x_1}\cdot\mathbb{1}_{p_1}}{\underset{\downarrow}{e^{-i\epsilon\mathscr{T}_0(p)}}}e^{-i\epsilon\mathcal{V}(x)}|x_0\rangle \tag{1.2.8}$$

acting with $\exp[-i\epsilon\mathscr{T}_0(p)]$ on $\langle p|$ and with $\exp[-i\epsilon\mathcal{V}(x)]$ on $|x\rangle$

$$=\lim_{\epsilon\to0}\int\left(\prod_{k=1}^{n}\frac{\mathrm{d}p_k}{(2\pi)^D}\right)\left(\prod_{k=1}^{n-1}\mathrm{d}x_k\right)e^{-i\epsilon\sum_{k=1}^{n}\left(\mathscr{T}_0(p_k)+\mathcal{V}(x_{k-1})\right)}\langle x_n|p_n\rangle\cdots\langle p_1|x_0\rangle$$

finally recalling that $\langle x|p\rangle=\langle p|x\rangle^*=\exp\big[i(p\cdot x)\big]$

$$=\lim_{\epsilon\to0}\int\left(\prod_{k=1}^{n}\frac{\mathrm{d}p_k}{(2\pi)^D}\right)\left(\prod_{k=1}^{n-1}\mathrm{d}x_k\right)e^{i\epsilon\sum_{k=1}^{n}\left[\frac{p_k\cdot(x_k-x_{k-1})}{\epsilon}-\left(\mathscr{T}_0(p_k)+\mathcal{V}(x_{k-1})\right)\right]} \quad . \tag{1.2.9}$$

We have obtained a result for the integral kernel $\mathcal{K}(x_f,t_f;x_i,t_i)$ which is formally equivalent to the one formulated by Feynman in (1.1.2), despite the fact that it is defined in phase space rather than in configuration space. However we will see that in most cases[3] the integral in the momenta can be easily performed reducing effectively (1.2.9) to (1.1.2).

[1] we will analyze later the case of more general Hamiltonian functions.

[2] this formula is discussed in Appendix A. However, as better explained in the appendix, it gives us a valid approximant only to first order, and for our particular needs it turns out to be sufficient; however if we wanted a more precise approximation (with respect to some parameter, perhaps the commutator itself), we had to use the Zassenhaus formula[8, 9]:

$$e^{\lambda(A+B)}=e^{\lambda A}e^{\lambda B}e^{\lambda^2\frac{1}{2}[B,A]}e^{\lambda^3\frac{1}{6}[[B,A],A+2B]}\cdots \quad .$$

See Appendix A of this part for further details.

[3] obviously, even in the particular case we are analyzing.

Our result, eq. (1.2.9), can be put in a much more elegant form effectively taking the limit $\epsilon \to 0$. If we consider the paths in phase space $x(t)$ and $p(t)$, such that $x(t_k) \equiv x_k$ and $p(t_k) \equiv p_k$, they define a functional space, and in the limit the argument of the exponential in (1.2.9) becomes just an integral over t[1]:

$$\begin{aligned}
&\epsilon \sum_{k=1}^{n} \left[\frac{p_k \cdot (x_k - x_{k-1})}{\epsilon} - \left(\mathscr{T}_0(p_k) + \mathcal{V}(x_{k-1}) \right) \right] \\
&= \epsilon \sum_{k=1}^{n} \left[p(t_k) \cdot \dot{x}(t_k) - \mathcal{H}\big(x(t_k), p(t_k)\big) \right] + O(\epsilon^2) \\
&\to \int_{t_i}^{t_f} \mathrm{d}t \left(p(t) \cdot \dot{x}(t) - \mathcal{H}\big(x(t), p(t)\big) \right) \quad . \qquad (1.2.10)
\end{aligned}$$

Furthermore, the integration over all the points $x(t_k)$ and $p(t_k)$ in the limit leads to a *functional integration* $\int \mathscr{D}[x(t)]\, \mathscr{D}[p(t)]$ over all the possible paths in phase space[2,3] that start from x_i at t_i and end at x_f at t_f. Eq. (1.2.9) then becomes an integral over all constrained paths $x(t)$, as well as over all unconstrained $p(t)$[4]:

Functional Integration

$$\mathcal{K}(x_f, t_f; x_i, t_i) = \int\limits_{\substack{x(t_i)=x_i \\ x(t_f)=x_f}} \mathscr{D}[x(t)]\, \mathscr{D}[p(t)]\, \exp\left\{ i \int_{t_i}^{t_f} \mathrm{d}t \left(p(t)\dot{x}(t) - \mathcal{H}\big(x(t), p(t)\big) \right) \right\} . \quad (1.2.11)$$

We discuss now the case of a general Hamiltonian operator of two variables x and p[5] (that can possibly contain mixed $x_i p_j$ terms)[6]. We would like to repeat the same procedure we have just done through Eq.s (1.2.5)– (1.2.11), but we cannot use Trotter formula (1.2.6) anymore, since it is difficult to treat exponentials of mixed terms $x_i p_j$, because we don't know how to act on them with $|p\rangle$ and $|x\rangle$, in order to extract the Hamiltonian function from the Hamiltonian operator. If it would be possible to manage $\exp\{i\epsilon\mathcal{H}\}$ to have all the ps to the left of all the xs[7], then we could repeat the procedure done in (1.2.8), and then continue in the same previous way to obtain (1.2.11)[8].The big problem is that $e^{\mathscr{O}(x,p)} \neq : e^{\mathscr{O}(x,p)} :$, in

[1] about $O(\epsilon^2)$ in the second line, refer to Appendix A for a more detailed discussion on errors and approximations in path integrals.

[2] this integration is intended to be defined by a regularization process, *e.g.*:

$$\int \mathscr{D}[x(t)]\, \mathscr{D}[p(t)] \equiv \lim_{\epsilon \to 0} \int \prod_{k=1}^{n} \frac{\mathrm{d}p_k}{(2\pi)^D} \prod_{k=1}^{n-1} \mathrm{d}x_k \quad \text{(time-slicing)}.$$

However time slicing is not the only regularization possible for this functional measure, as we will see in the following.

[3] There is in fact a mathematical notion of integration over paths due to Wiener[10, 11], but unfortunately this notion *cannot* be used in this case (we need to perform an analytic continuation in time, the *Wick rotation*, in order to use Wiener measure).

[4] that is why it is called *path integral*.

[5] in this general formulation, the variables x and p are two complete sets of independent observables, maybe *not* the ones related to position and momentum coordinates.

[6] that doesn't represent a big deal as far as we limit ourselves to order ϵ in the approximations, as it is done in most cases, however it becomes tougher to use more precise approximations (see Appendix A).

[7] we call such an operator ordering *normal ordering*, and it will be denoted by $: \ldots :$, *i.e.* given an arbitrary operator $\mathscr{O}(x,p)$, we will denote with $: \mathscr{O}(x,p) :$ its normal ordered form, with all ps to the left of all xs.

[8] considering the whole $: \exp\{i\epsilon\mathcal{H}\} :$ instead of $\exp\{i\epsilon\mathscr{T}_0\} \exp\{i\epsilon\mathcal{V}\}$.

particular either $e^{:\mathcal{O}(x,p):} \neq :e^{:\mathcal{O}(x,p):}:$! But in the latter case, we are able to write a relation between the two different exponentials. In particular if the exponential depends on some parameter (*e.g.* ϵ in $\exp\{-i\epsilon\mathcal{H}\}$), the relation will be a power series of such parameter.

So when we are dealing with a general time-independent Hamiltonian, all we have to do is to put it in *normal ordered form* $\mathcal{H}_{\mathcal{N}}(x,p)$, and use the following relation[12]:

$$e^{-i\epsilon\mathcal{H}_{\mathcal{N}}(x,p)} =: e^{-i\epsilon\mathcal{H}_{\mathcal{N}}(x,p)} : -\epsilon^2 \sum_{n=0}^{\infty} \frac{(-i\epsilon)^n}{(n+2)!}\left(\mathcal{H}_{\mathcal{N}}^{n+2}(x,p) - :\mathcal{H}_{\mathcal{N}}^{n+2}(x,p):\right) \quad ; \qquad (1.2.12)$$

finally, since we need to consider only terms up to order ϵ in the limit $\epsilon \to 0$, one can proceed through Eq.s (1.2.8)–(1.2.9) using $:e^{-i\epsilon\mathcal{H}_{\mathcal{N}}(x,p)}:$ to obtain

$$\mathcal{K}(x_f,t_f;x_i,t_i) = \int_{\substack{x(t_i)=x_i \\ x(t_f)=x_f}} \mathscr{D}[x(t)]\,\mathscr{D}[p(t)]\,\exp\left\{ i\int_{t_i}^{t_f} \mathrm{d}t\,\Big(p(t)\dot{x}(t) - \mathcal{H}_{\mathcal{N}}\big(x(t),p(t)\big)\Big)\right\}. \qquad (1.2.13)$$

Path Integral for time-independent Hamiltonians

1.2.2: *Time-dependent Hamiltonian*

The most general Hamiltonian function could bear an explicit time dependence, so it would be not a constant of the motion anymore, since $\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial t}$. Nevertheless we know that evolution operator obeys Dyson series (1.2.3), and we will show that a path integral expression of the propagator is still possible[1]. Consider an Hamiltonian function in normal ordering $\mathcal{H}(x,p,t)$ bearing an explicit time dependence. The integral kernel we have to calculate has the form:

$$\mathcal{K}(x_f,t_f;x_i,t_i) = \langle x_f|\mathcal{U}(t_f,t_i)|x_i\rangle \quad . \qquad (1.2.14)$$

Up to this point we need to describe the two-parameter family of unitary operators $\mathcal{U}(t,s)$ a little more in depth. We know that for a time-independent Hamiltonian it reduces to a one-parameter family of unitary operators that satisfies the properties of a group, and being time its parameter it turns out to be the *unitary representation* of time translations group. We would like to know the properties of the two-parameter family $\mathcal{U}(t,s)$ as well.

Definition 1.2.1 (Unitary Propagator)**.** A two-parameter family of unitary operators $\mathcal{A}(s,t)$, s , $t \in \mathbb{R}$ which satisfies:

(i) $\mathcal{A}(r,s)\mathcal{A}(s,t) = \mathcal{A}(r,t)$

(ii) $\mathcal{A}(t,t) = \mathbb{1}$

(iii) $\mathcal{A}(s,t)$ is jointly strongly continuous in s and t

is called a **unitary propagator**.

[1] this derivation follows the one given in [5].

Theorem 1.2.1 (the Dyson expansion[1]). *Let $t \to \mathcal{H}(t)$ be a strongly continuous map of $\mathbb{R}$ into the* bounded *self-adjoint operators on a Hilbert space $\mathscr{H}$. Then there is a unitary propagator $\mathcal{U}(t,s)$ on $\mathscr{H}$ so that, for all $\psi \in \mathscr{H}$,*

$$\varphi_s(t) = \mathcal{U}(t,s)\psi$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_s(t) = -i\mathcal{H}(t)\varphi_s(t)\,, \quad \varphi_s(s) = \psi \quad .$$

Unfortunately in most cases the Hamiltonian $\mathcal{H}(x,p,t)$ does not satisfy the conditions of Theorem 1.2.1, since it often has an unbounded kinetic part. Nevertheless time dependent Hamiltonians physically relevant are usually of the form $\mathcal{H}(t) = H_0 + V(t)$, where both H_0 and $V(t)$ are normal ordered operators depending on coordinates and momenta, and H_0 is possibly unbounded, while $V(t)$ satisfies the conditions of Theorem 1.2.1. In such a case we are able to express $\mathcal{K}(x_f,t_f;x_i,t_i)$ in a functional integral form[2]. In fact we can use the last theorem, passing to the "interaction representation". We define

Evolution in a time-dependent system

$$\widetilde{V}(t) = e^{iH_0t}V(t)e^{-iH_0t} \quad ;$$

then $t \to \widetilde{V}(t)$ also satisfies the hypotheses of Theorem 1.2.1, and we call the corresponding propagator $\widetilde{\mathcal{U}}(t,s)$. We recall that it is determined by a Dyson series (see Appendix B for a proof):

$$\widetilde{\mathcal{U}}(t,s) = \mathbb{1} + \sum_{n=1}^{\infty}(-i)^n \int_s^t \int_s^{t_1} \cdots \int_s^{t_{n-1}} \mathrm{d}t_1 \cdots \mathrm{d}t_n\, e^{iH_0t_1}V(t_1)\cdots e^{-iH_0(t_{n-1}-t_n)}V(t_n)e^{-iH_0t_n} \quad . \tag{1.2.15}$$

Now setting

$$\mathcal{U}(t,s) = e^{-itH_0}\widetilde{\mathcal{U}}(t,s)e^{isH_0} \quad , \tag{1.2.16}$$

we see that, at least formally, $\mathcal{U}(t,s)$ satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}(t,s) &= -iH_0e^{-itH_0}\widetilde{\mathcal{U}}(t,s)e^{isH_0} + e^{-itH_0}\Big(-i\widetilde{V}(t)\Big)\widetilde{\mathcal{U}}(t,s)e^{isH_0} \\ &= -i\Big(H_0 + V(t)\Big)\mathcal{U}(t,s) \quad ,\end{aligned}$$

so $\varphi_s(t) = \mathcal{U}(t,s)\psi$ should be a strong solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_s(t) = -i\Big(H_0 + V(t)\Big)\varphi_s(t)\,,\ \varphi_s(s) = \psi \quad .$$

[1] for a proof, and a discussion on its implications see Appendix B.
[2] see Appendix B for the most general case.

The problem arises from the fact that $H_0\,\mathcal{U}(t,s)\psi$ may not take sense, since $\widetilde{\mathcal{U}}(t,s)\psi$ may not be in the domain of H_0 even if ψ is. Now a particular case of the general Theorem B.1.2 of Appendix B assures us that if $t \to [H_0, V(t)]$ is strongly continuous, then $\varphi_s(t)$ is a strong solution; however it is always a "weak" solution, in the sense that for any $\eta \in D(H_0)$, $(\eta, \varphi_s(t))$[1] is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\eta, \varphi_s(t)) = -i(H_0\eta, \varphi_s(t)) - i(V(t)\eta, \varphi_s(t)) \quad .$$

Now the evolution operator $\mathcal{U}(t_f, t_i)$, if exists, is indeed a *unitary propagator*, since $\widetilde{\mathcal{U}}(t,s)$ is a unitary propagator. Property (i) in particular turns out to be very useful since it allows us to write, defining $t_f \equiv t_n$, $t_i \equiv t_0$:

$$\mathcal{U}(t_n, t_0) = \mathcal{U}(t_n, t_{n-1}) \cdots \mathcal{U}(t_1, t_0) \tag{1.2.17}$$

$$\text{with } t_k \equiv t_{k+1} - \epsilon \ , \ \epsilon = \frac{t_f - t_i}{n} \text{ so obviously } t_n > t_{n-1} > \ldots > t_0 \ .$$

So Eq. (1.2.14) becomes:

$$\mathcal{K}(x_n, t_n; x_0, t_0) = \langle x_n | \mathcal{U}(t_n, t_{n-1}) \cdots \mathcal{U}(t_1, t_0) | x_0 \rangle \quad . \tag{1.2.18}$$

Now using the decompositions of the identity operator $\mathbb{1}_x$ and $\mathbb{1}_p$ defined in section 1.2.1 we have:

$$\begin{aligned}\mathcal{K}(x_n, t_n; x_0, t_0) &= \langle x_n | \underset{\downarrow \mathbb{1}_{p_n}}{\mathcal{U}(t_n, t_{n-1})} \quad \cdots \quad \underset{\mathbb{1}_{x_{n-1}} \cdot \downarrow \mathbb{1}_{p_{n-1}}}{\mathcal{U}(t_1, t_0)} \underset{\mathbb{1}_{x_1} \cdot \downarrow \mathbb{1}_{p_1}}{} | x_0 \rangle \\ &= \int \left(\prod_{k=1}^{n} \frac{\mathrm{d}p_k}{(2\pi)^D} \right) \left(\prod_{k=1}^{n-1} \mathrm{d}x_k \right) \langle x_n | p_n \rangle \langle p_n | \mathcal{U}(t_n, t_{n-1}) | x_{n-1} \rangle \cdots \langle p_1 | \mathcal{U}(t_1, t_0) | x_0 \rangle \quad .\end{aligned} \tag{1.2.19}$$

All we have to do is evaluate $\langle p_k | \mathcal{U}(t_k, t_{k-1}) | x_{k-1} \rangle$, and put it back in (1.2.19). But expanding t_k around t_{k-1} in Eq. (1.2.15) we obtain:

$$\widetilde{\mathcal{U}}(t_k, t_{k-1}) = \mathbb{1} - i\epsilon \, e^{iH_0 t_{k-1}} V(t_{k-1}) e^{-iH_0 t_{k-1}} + O(\epsilon^2) \quad ;$$

then

$$\begin{aligned}\mathcal{U}(t_k, t_{k-1}) &= e^{-it_k H_0} \, \widetilde{\mathcal{U}}(t_k, t_{k-1}) e^{it_{k-1} H_0} \\ &= e^{-i(t_k - t_{k-1})H_0} \Big(\mathbb{1} - i\epsilon V(t_{k-1}) \Big) + O(\epsilon^2) \\ &= e^{-i\epsilon H_0} e^{-i\epsilon V(t_{k-1})} + O(\epsilon^2) = e^{-i\epsilon \mathcal{H}(t_{k-1})} + O(\epsilon^2) \\ &=: e^{-i\epsilon \mathcal{H}(t_{k-1})} : + O(\epsilon^2) \quad ,\end{aligned}$$

recalling Eq. (1.2.12) in the last equality. So if we take the limit $\epsilon \to 0$ we can forget about second-order ϵ-terms, and acting with $\langle p_k|$ and $|x_{k-1}\rangle$ on the normal ordered exponential we get:

$$\lim_{\epsilon \to 0} \langle p_k | \mathcal{U}(t_k, t_{k-1}) | x_{k-1} \rangle = \lim_{\epsilon \to 0} e^{-i\epsilon \mathcal{H}(x_{k-1}, p_k, t_{k-1})} \langle p_k | x_{k-1} \rangle \quad ; \tag{1.2.20}$$

[1] $D(H_0)$ is the domain of H_0 and $(\eta, \varphi_s(t))$ is the scalar product between $\eta, \varphi_s(t) \in \mathscr{H}$.

finally inserting this result in (1.2.19) we obtain the discretized path integral:

$$\mathcal{K}(x_n,t_n;x_0,t_0)=\lim_{\epsilon\to 0}\int\left(\prod_{k=1}^{n}\frac{\mathrm{d}p_k}{(2\pi)^D}\right)\left(\prod_{k=1}^{n-1}\mathrm{d}x_k\right)e^{i\epsilon\sum\limits_{k=1}^{n}\left[\frac{p_k\cdot(x_k-x_{k-1})}{\epsilon}-\mathcal{H}(x_{k-1},p_k,t_{k-1})\right]}\ . \quad (1.2.21)$$

Following the same reasoning of (1.2.10), we obtain the final path integral formula:

Functional Integral for time-dependent Hamiltonians

$$\mathcal{K}(x_f,t_f;x_i,t_i)=\int\limits_{\substack{x(t_i)=x_i\\x(t_f)=x_f}}\mathscr{D}[x(t)]\,\mathscr{D}[p(t)]\,\exp\left\{i\int_{t_i}^{t_f}\mathrm{d}t\,\Big(p(t)\dot{x}(t)-\mathcal{H}\big(x(t),p(t),t\big)\Big)\right\}. \quad (1.2.22)$$

We see that this formula is perfectly analogous to Eq. (1.2.11), provided with the right time dependence of the Hamiltonian function.

1.2.3: *Transition to Lagrangian formulation*

We see that in Equations (1.2.11), (1.2.13) and (1.2.22) the integrand in the exponential looks like the Lagrangian function $\mathscr{L}$ associated with the Hamiltonian $\mathcal{H}$; path integral thus becoming the integral over paths of the exponential of i[1] times the classical action associated with the system. But this appearance is misleading, since the momenta $p(t)$ are independent variables, not yet related to $x(t)$ or their derivatives. However, if the Hamiltonian is a quadratic function of the $p(t)$, then the integral over this variables can be performed just substituting them with their values dictated by canonical formalism, thus the integrand in the exponential really is the Lagrangian function[13].

Let's see how this passage is performed: we deal with a quadratic form of the $p(t)$, *i.e.*[2]

$$\mathcal{H}(x(t),p(t),t)=\frac{1}{2}A^{ij}[x(t),t]p_i(t)p_j(t)+B^i[x(t),t]p_i(t)+C[x(t),t]\quad ;$$

so the exponential becomes

$$\exp\left\{-i\Big(\frac{1}{2}\mathscr{A}^{ij}p_ip_j+\mathscr{B}^ip_i+\mathscr{C}\Big)\right\}\quad ,$$

where we have used a shorthand notation where an index like ^{i} stands for a multi-index, both discrete and continuous, and the contraction of equal indices stands for summation over discrete indices and integration over continuous ones[3] and

$$\begin{aligned}\mathscr{A}^{ij}&\equiv A^{ij}[x(t),t]\delta(t-t')\quad ,\\ \mathscr{B}^i&\equiv B^i[x(t),t]-\dot{x}^i(t)\quad ,\\ \mathscr{C}&\equiv\int\mathrm{d}t\,C[x(t),t]\quad .\end{aligned}$$

[1] (times $\hbar^{-1}$).
[2] making explicit the components of the $x(t)$ and $p(t)$ vectors, and using Einstein summation convention.
[3] see Appendix C for further informations.

Now we know how to solve Gaussian integrals like $\int \mathscr{D}[p(t)] \exp\left\{-i\left(\frac{1}{2}\mathscr{A}^{ij}p_ip_j + \mathscr{B}^i p_i + \mathscr{C}\right)\right\}$[1] : it is sufficient to substitute to the variables p_i their value $\bar{p}_i \equiv -\mathscr{A}^{-1}_{ij}\mathscr{B}^j$, *i.e.* the stationary point of the exponent[2]. But the stationary point of $\int \mathrm{d}t \left(p(t)\dot{x}(t) - \mathcal{H}\big(x(t), p(t), t\big)\right)$ obeys the equation:

$$\dot{x}^i(t) = \left.\frac{\delta \mathcal{H}\big(x(t), p(t), t\big)}{\delta p_i(t)}\right|_{p=\bar{p}} \quad ,$$

i.e. the usual formula dictated by canonical formalism. So performing the integration over momentum paths we see that the exponent of Equations (1.2.11) , (1.2.13) and (1.2.22) is indeed i times the classical action of the system; therefore finally we can write the path integral in configuration space[3]:

$$\mathcal{K}(x_f, t_f; x_i, t_i) = \int\limits_{\substack{x(t_i)=x_i \\ x(t_f)=x_f}} \mathscr{D}[x(t)] \exp\left\{i \int_{t_i}^{t_f} \mathrm{d}t\, \mathscr{L}\big(x(t), \dot{x}(t), t\big)\right\}. \tag{1.2.23}$$

Path Integral in Lagrangian formulation

However this derivation is just formal and does not take into account the factor

$$\frac{1}{\sqrt{\mathrm{Det}\left[\frac{i\mathscr{A}[x(t)]}{2\pi}\right]}} \quad ,$$

that contains a functional determinant that has to be suitably managed. In particular if $\mathscr{A}$ does not depend on $x(t)$, such a determinant could be adsorbed in the definition of the measure $\mathscr{D}[x(t)]$, however if it depends on $x(t)$ could give rise to modifications to the Lagrangian $\mathscr{L}$ that one has to take into account, in particular we will see that in curved space-time the integration of momenta modifies $\mathscr{L}$ introducing a counterterm that depends on the regularization scheme we have chosen to properly define the functional measure $\mathscr{D}[x(t)]\mathscr{D}[p(t)]$.

Integrating out momenta is a touchy business

1.3: *Applications: Free Theories, Harmonic Oscillators & Perturbative Theory*

It is now time to explicit our general results in some useful cases. Let's start from the simplest case: a system consisting of n free[4] particles of equal mass m in a N dimensional

[1] see Appendix C

[2] the value $\bar{p}$ obeying:

$$\left.\frac{\delta}{\delta p_i}\left(-i\left(\frac{1}{2}\mathscr{A}^{ij}p_ip_j + \mathscr{B}^i p_i + \mathscr{C}\right)\right)\right|_{p=\bar{p}} = 0 \quad ;$$

where the derivative is a functional derivative.

[3] we present here only the most general case of a time-dependent Lagrangian function, the other are particular cases.

[4] as free particles we mean distinguishable particles that don't interact neither mutually, neither with some external potential.

space. In that case we can describe our system with $D \equiv nN$ coordinates and momenta *i.e.* with a D-dimensional configuration space described by a Lagrangian function

$$\mathscr{L}_{free}\big(x(t), \dot{x}(t)\big) = \frac{m}{2}\dot{x}(t)^2 \tag{1.3.1}$$

where we use a shorthand notation for a D-dimensional vector

$$x(t)^2 \equiv \sum_{i=1}^{D} x^i(t)^2 \quad ,$$

and the dot stands for a derivative with respect to time as usual.

Before dealing with the path integral of such a system we will see that if the Lagrangian function is a quadratic form, the kernel $\mathcal{K}(x_f, t_f; x_i, t_i)$ is proportional to the exponential of i times the action evaluated on the classical path, *i.e.* on the path satisfying the classical equations of motion and the boundary conditions[1]: consider a general quadratic Lagrangian function

$$\mathscr{L} = \frac{1}{2}(a_{ij}\dot{x}^i\dot{x}^j + b_{ij}x^i x^j) + c_{ij}x^i\dot{x}^j + d_i\dot{x}^i + e_i x^i \quad ,$$

where the implicit time dependence of coordinates and their time derivatives has been omitted, and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are $D \times D$ dimensional time-dependent matrices and d, e are D-dimensional time-dependent vectors, in the usual component notation with summation over repeated indices. Now it is immediate to see that $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$, *i.e.* both $\mathbf{a}$ and $\mathbf{b}$ are symmetric matrices[2]. With matrix $\mathbf{c}$ we have to be more careful: a priori it has not a definite symmetry, but as usual we can split it in a symmetric and an antisymmetric part defining

$$c_{ij} = c^s_{ij} + c^a_{ij} \quad ,$$

with

$$c^s_{ij} \equiv \frac{c_{ij} + c_{ji}}{2} \quad ,$$
$$c^a_{ij} \equiv \frac{c_{ij} - c_{ji}}{2} \quad ;$$

but the term $c^s_{ij}x^i\dot{x}^j$ can be written as a total derivative (that we can cancel out of the Lagrangian, not contributing to the equations of motion) plus a term with the structure $\alpha_{ij}x^i x^j$ that can be adsorbed redefining b_{ij} in a suitable way:

$$c^s_{ij}x^i\dot{x}^j = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}c^s_{ij}x^i x^j\right) - \frac{1}{2}\dot{c}^s_{ij}x^i x^j \quad .$$

[1] the action evaluated on the classical path is not to be confused with the classical action: the latter is the action of a classical system corresponding to the quantum mechanical Hamiltonian function of the system under examination; the former is the classical action evaluated on the path dictated by classical equations of motion and boundary conditions.

[2] in fact if they have not a definite symmetry, they can be split in a symmetric and antisymmetric part, but the antisymmetric part contracted with the product of two coordinates (that is symmetric) vanishes.

In an analogous way we can see that the term $d_i\dot{x}^i$ equals a total derivative plus a term adsorbed in redefining vector e; so finally a general quadratic Lagrangian can be written as

$$\mathscr{L} = \frac{1}{2}(a^s_{ij}\dot{x}^i\dot{x}^j + b^s_{ij}x^ix^j) + c^a_{ij}x^i\dot{x}^j + e_ix^i \quad , \tag{1.3.2}$$

where the symmetry or antisymmetry of the matrices has been made explicit with indices ^{s} and ^{a} that will be henceforward omitted. Equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{x}^i}\right) = \frac{\partial\mathscr{L}}{\partial x^i}$$

read in that case

$$\dot{a}_{ij}x^j + a_{ij}\ddot{x}^j - \dot{c}_{ij}x^j - 2c_{ij}\dot{x}^j - b_{ij}x^j - d^i = 0 \quad . \tag{1.3.3}$$

If we now perform a split $x(t) = x_{cl}(t) + q(t)$, where $x_{cl}(t)$ is a vector satisfying equations of motion and the boundary conditions, *i.e.*[1]:

$$\begin{aligned} &x_{cl}(t_i) = x_i \ , \ x_{cl}(t_f) = x_f \quad , \\ &\dot{a}_{ij}x^j_{cl} + a_{ij}\ddot{x}^j_{cl} - \dot{c}_{ij}x^j_{cl} - 2c_{ij}\dot{x}^j_{cl} - b_{ij}x^j_{cl} - d^i = 0 \quad : \end{aligned}$$

Classical Path & Quantum Fluctuations split

and $q(t)$ has vanishing boundary conditions, *i.e.*[2]:

$$q(t_i) = 0 \ , \ q(t_f) = 0 \quad ;$$

we immediately see that the Lagrangian can be split as:

$$\mathscr{L}(x,\dot{x}) \equiv \mathscr{L}(x_{cl},\dot{x}_{cl}) + \mathscr{L}_2(q,\dot{q}) + \left(a_{ij}\dot{q}^i\dot{x}^j_{cl} + b_{ij}q^ix^j_{cl} - c_{ij}\dot{q}^ix^j_{cl} + c_{ij}q^i\dot{x}^j_{cl} + d_iq^i\right) \quad ,$$

where $\mathscr{L}_2(q,\dot{q}) \equiv \mathscr{L}(q,\dot{q}) - d_iq^i$ (it is the purely quadratic part of the Lagrangian). In order to evaluate the kernel $\mathcal{K}(x_f,t_f;x_i,t_i)$ we need the action $S[x,t_f,t_i]$, *i.e.* the time integral between the initial and final time of the Lagrangian function. But $S[x,t_f,t_i] \equiv S[x_{cl},t_f,t_i] + S_2[q,t_f,t_i]$ where

$$S_2[q,t_f,t_i] = \int_{t_i}^{t_f}\mathrm{d}t\,\mathscr{L}_2(q,\dot{q}) \quad :$$

in fact the time integral

$$\int_{t_i}^{t_f}\mathrm{d}t\left(a_{ij}\dot{q}^i\dot{x}^j_{cl} + b_{ij}q^ix^j_{cl} - c_{ij}\dot{q}^ix^j_{cl} + c_{ij}q^i\dot{x}^j_{cl} + d_iq^i\right)$$

vanishes. In order to prove the last assertion is sufficient to integrate by parts the $\dot{q}$-terms, remembering that the q function vanishes at the boundaries and x_{cl} satisfies equations of

[1] $x_{cl}(t)$ is often called the "classical path".
[2] $q(t)$ is often called the "quantum fluctuation".

motion. Finally since we usually define the functional measure $\mathscr{D}[x(t)]$ to be invariant under translations, so that $\mathscr{D}[x(t)] \equiv \mathscr{D}[q(t)]$, formula (1.2.23) becomes in that case:

$$\mathcal{K}(x_f,t_f;x_i,t_i) = e^{iS[x_{cl},t_f,t_i]} \int_{VBC} \mathscr{D}[q(t)]\ \exp\left\{ i\int_{t_i}^{t_f} \mathrm{d}t\, \mathscr{L}_2\big(q(t),\dot{q}(t),t\big)\right\} \quad , \tag{1.3.4}$$

where VBC stands for vanishing boundary conditions. Now defining

$$\int_{VBC} \mathscr{D}[q(t)]\ \exp\left\{ i\int_{t_i}^{t_f} \mathrm{d}t\, \mathscr{L}_2\big(q(t),\dot{q}(t),t\big)\right\} \equiv \mathcal{F}(t_f,t_i) \quad ,$$

we see explicitly that the kernel is proportional to the exponential of the action on the classical path:

$$\mathcal{K}(x_f,t_f;x_i,t_i) = \mathcal{F}(t_f,t_i)\ e^{iS[x_{cl},t_f,t_i]} \quad . \tag{1.3.5}$$

1.3.1: *Systems of free particles*

We will now calculate explicitly $\mathcal{K}(x_f,t_f;x_i,t_i)$ in the case of a system of free particles. We have seen at the beginning of this section that such a system has in general a Lagrangian function:

$$\mathscr{L}_{free}\big(x(t),\dot{x}(t)\big) = \frac{m}{2}\dot{x}(t)^2 \tag{1.3.6}$$

where $x(t)$ is a D-dimensional vector. We have to make a sense out of the functional measure $\mathscr{D}[x(t)]$, and in that simple case it is useful to remember the time discretization that led to our path integral formulation, so we will write

$$\mathscr{D}[x(t)] \equiv \lim_{\epsilon\to 0}\Big(\frac{m}{2\pi i\epsilon}\Big)^{\frac{D}{2}N} \prod_{i=1}^{N-1} \mathrm{d}x_i \quad ,$$

where $\mathrm{d}x_i$ is the D dimensional Lebesgue measure, $\epsilon \equiv \frac{t_f-t_i}{N}$ and the constant multiplying the measures is due to the integration over momenta (see section 1.2.3). The time discretized action becomes $\epsilon\sum \frac{m}{2}(\frac{x_i-x_{i-1}}{\epsilon})^2$, so we have to evaluate the following expression[1]:

$$\mathcal{K}(x_f,t_f;x_i,t_i) = \lim_{\epsilon\to 0}\int \prod_{i=1}^{N-1} \mathrm{d}x_i \Big(\frac{m}{2\pi i\epsilon}\Big)^{\frac{D}{2}N} \exp\left\{ i\epsilon\sum_{i=1}^{N}\frac{m}{2}\Big(\frac{x_i-x_{i-1}}{\epsilon}\Big)^2\right\} \quad . \tag{1.3.7}$$

But if we define

$$F_\epsilon(x) \equiv \Big(\frac{m}{2\pi i\epsilon}\Big)^{\frac{D}{2}} \exp\left\{\frac{imx^2}{2\epsilon}\right\} \quad ,$$

we see that with the usual convolution product[2]

$$\big(F_\epsilon * F_{\epsilon'}\big)(x) \equiv \int_{-\infty}^{\infty} \mathrm{d}y\, F_\epsilon(y) F_{\epsilon'}(x-y)$$

[1] identifying $x_N \equiv x_f$ and $x_0 \equiv x_i$.

[2] being the Lebesgue measure translationally invariant, this definition is equivalent to:

$$\big(F_\epsilon * F_{\epsilon'}\big)(x) \equiv \int_{-\infty}^{\infty} \mathrm{d}y\, F_\epsilon(y-a) F_{\epsilon'}(x-y) \quad ,$$

where a is an arbitrary factor.

holds the following property:

$$\Big(F_\epsilon * F_{\epsilon'}\Big)(x) = F_{\epsilon+\epsilon'}(x) \quad . \qquad \text{(first convolution property)}$$

The demonstration of the (first convolution property) involves some boring calculations completing squares and performing a Gaussian integration on y[1]. The usefulness of this property is seen considering the integration over $\mathrm{d}x_i$ in Eq. (1.3.7), isolating x_i-depending terms we see that this integration reduces to:

$$\Big(\frac{m}{2\pi i\epsilon}\Big)^{\frac{D}{2}}\Big(\frac{m}{2\pi i\epsilon}\Big)^{\frac{D}{2}}\int \mathrm{d}x_i\, \exp\Big\{\frac{im(x_i-x_{i-1})^2}{2\epsilon}\Big\}\exp\Big\{\frac{im(x_{i+1}-x_i)^2}{2\epsilon}\Big\} =$$

remembering the definition of convolution product

$$= \Big(F_\epsilon * F_\epsilon\Big)(x_{i+1}-x_{i-1}) = F_{2\epsilon}(x_{i+1}-x_{i-1}) \quad .$$

Now the integration over $\mathrm{d}x_{i+1}$ is perfectly analogous but involves the convolution product $\Big(F_{2\epsilon} * F_\epsilon\Big)(x_{i+2}-x_{i-1})$; so it is straightforward to perform all consecutive $N-1$ integrations, starting from $\int \mathrm{d}x_1$, obtaining[2]:

$$F_{N\epsilon}(x_N - x_0) \equiv \Big(\frac{m}{2\pi i(t_f-t_i)}\Big)^{\frac{D}{2}}\exp\Big\{\frac{im(x_f-x_i)^2}{2(t_f-t_i)}\Big\} \quad ;$$

so finally

$$\mathcal{K}_{free}(x_f,t_f;x_i,t_i) = \Big(\frac{m}{2\pi i(t_f-t_i)}\Big)^{\frac{D}{2}}\exp\Big\{\frac{im(x_f-x_i)^2}{2(t_f-t_i)}\Big\} \quad . \qquad (1.3.8)$$

As we expected, the Kernel in Eq. (1.3.8) has the structure predicted in (1.3.5), being the free Lagrangian effectively a quadratic form[3]. Furthermore, we could have found $\mathcal{K}_{free}$ starting from Eq. (1.3.5): we know that $\mathcal{K}_{free}(x_f,t_f;x_i,t_i) = \mathcal{F}(t_f,t_i)\ \exp\Big\{iS[x_{cl},t_f,t_i]\Big\}$, so all we need to evaluate is $\mathcal{F}(t_f,t_i)$; but since Eq. (1.2.1) tells us that $\mathcal{K}$ obeys Schrödinger

[1] for examples on such a method as "completing squares" in Gaussian integration see Appendix C.

[2] remembering that $x_N \equiv x_f$, $x_0 \equiv x_i$ and $N\epsilon \equiv t_f - t_i$.

[3] the solution of equations of motion

$$\ddot{x} = 0$$

respecting the boundary conditions $x(t_i) = x_i$ and $x(t_f) = x_f$ is

$$x(t) = \frac{x_f - x_i}{t_f - t_i}t + \frac{x_i t_f - x_f t_i}{t_f - t_i} \quad ,$$

so the action on the classical path is effectively

$$S[x_{cl},t_f,t_i] = \frac{m}{2}\frac{(x_f-x_i)^2}{(t_f-t_i)} \quad .$$

Equation[1], we can fix $\mathcal{F}(t_f, t_i)$ to be the value for which such equation is satisfied by $\mathcal{K}_{free}$. A straightforward calculation fixes obviously:

$$\mathcal{F}(t_f, t_i) = \left(\frac{m}{2\pi i(t_f - t_i)}\right)^{\frac{D}{2}} \quad ,$$

as we expected.

1.3.2: *The Harmonic Oscillator*

Another system of physical interest is a system of n uncoupled and distinguishable Harmonic Oscillators of equal mass m, in N dimensions, that is characterized in a $D = nN$ dimensional space by a Lagrangian function

$$\mathscr{L}_{osc}(x(t), \dot{x}(t)) = \frac{m}{2}\left(\dot{x}(t)^2 - \omega^2 x(t)^2\right) \quad ,$$

with usual vectorial notations. It is a discomforting task to evaluate directly $\mathcal{K}(x_f, t_f; x_i, t_i)$, since the path satisfying equations of motion and these general boundary conditions is:

$$x_{cl}(t) = \frac{\frac{x_f}{\cos(\omega t_f)}\tan(\omega t_i) + \frac{x_i}{\cos(\omega t_i)}\tan(\omega t_f)}{\tan(\omega t_f) - \tan(\omega t_i)}\cos(\omega t) + \frac{\frac{x_f}{\cos(\omega t_f)} - \frac{x_i}{\cos(\omega t_i)}}{\tan(\omega t_f) - \tan(\omega t_i)}\sin(\omega t) \quad ;$$

but since m and ω are constants, the Lagrangian function doesn't depend explicitly on time, so time translations are a symmetry of the system, and therefore $\mathcal{K}(x_f, t_f; x_i, t_i) \equiv \mathcal{K}\big(x_f, (t_f - t_i); x_i, 0\big)$, with the advantage that with these boundaries $x_{cl}(t)$ reduces to[2]:

$$x_{cl}(t) = x_i \cos(\omega t) + \frac{x_f - x_i \cos\big(\omega(t_f - t_i)\big)}{\sin\big(\omega(t_f - t_i)\big)}\sin(\omega t) \quad . \tag{1.3.9}$$

The action evaluated on Eq. (1.3.9) is

$$S_{osc}[x_{cl}, (t_f - t_i)] = \frac{m\omega}{2\sin\big(\omega(t_f - t_i)\big)}\Big((x_f^2 + x_i^2)\cos\big(\omega(t_f - t_i)\big) - 2x_f \cdot x_i\Big) \quad . \tag{1.3.10}$$

Since as the free Lagrangian function, $\mathscr{L}_{osc}$ is a quadratic form, we can use formula (1.3.5) and evaluate only

$$\mathcal{F}(t_f - t_i) = \int_{VBC} \mathscr{D}[q(t)] \exp\left\{i\frac{m}{2}\int_0^{(t_f - t_i)} \mathrm{d}t\,\big(\dot{q}(t)^2 - \omega^2 q(t)^2\big)\right\} \quad .$$

In order to do that we will use a particular technique, called "mode expansion": the idea is quite simple, we will expand paths $q(t)$ in a Fourier series, and define the integration over all possible paths as the integration of the Fourier coefficients over all their possible values,

[1] since $\mathcal{K}(x_f, t_f; x_i, t_i)$ is the only x_f and t_f-dependent object on the right member of Eq. (1.2.1), and Schrödinger Equation has to be satisfied for $\psi(x_f, t_f)$.

[2] we will limit ourselves to the case $0 < \omega(t_f - t_i) < \pi$ in order to determine $x_{cl}(t)$ univocally.

since every path vanishing at the boundaries could be written as a series of sines with real Fourier coefficients[1]. So we perform the expansion

$$q(t) = \sum_{n=1}^{\infty} C_n \phi_n(t) \quad ,$$

where $C_n \in \mathbb{R}^D$ and $\phi_n(t)$ is a complete set of orthonormal functions on $[0, (t_f - t_i)]$, vanishing at the boundaries, *i.e.* the sine functions

$$\phi_n(t) = \sqrt{\frac{2}{t_f - t_i}} \sin\left(\frac{n\pi t}{t_f - t_i}\right) \quad .$$

A simple integration by parts allow us to rewrite the action $S_2[q, (t_f - t_i)]$ as a function of the Fourier coefficients, precisely

$$S_2[q, (t_f - t_i)] \equiv \frac{m}{2} \int_0^{(t_f - t_i)} \mathrm{d}t \left(\dot{q}(t)^2 - \omega^2 q(t)^2\right) = \frac{m}{2} \sum_{n=1}^{\infty} C_n^2 \left[\left(\frac{n\pi}{t_f - t_i}\right)^2 - \omega^2\right] \quad .$$

So, setting

$$\mathscr{D}[q(t)] = \mathcal{M} \lim_{N\to\infty} \prod_{n=1}^{N} \mathrm{d}C_n \quad ,$$

we need to calculate

$$\mathcal{F}(t_f - t_i) = \mathcal{M} \lim_{N\to\infty} \int \prod_{n=1}^{N} \mathrm{d}C_n \, \exp\left\{ i \frac{m}{2} \sum_{n=1}^{N} C_n^2 \left[\left(\frac{n\pi}{t_f - t_i}\right)^2 - \omega^2\right]\right\} \quad ,$$

after performing N Gaussian integrations we obtain

$$\mathcal{F}(t_f - t_i) = \mathcal{M}' \lim_{N\to\infty} \prod_{n=1}^{N} \left[1 - \left(\frac{\omega(t_f - t_i)}{n\pi}\right)^2\right]^{-\frac{D}{2}} = \mathcal{M}' \left(\frac{\omega(t_f - t_i)}{\sin\left(\omega(t_f - t_i)\right)}\right)^{\frac{D}{2}} \quad ;$$

where $\mathcal{M}' \equiv \frac{t_f - t_i}{n\pi} \left(\frac{2\pi i}{m}\right)^{\frac{ND}{2}} \mathcal{M}$ and we have used the formula

$$\frac{\sin x}{x} = \lim_{N\to\infty} \prod_{n=1}^{N} \left[1 - \left(\frac{x}{n\pi}\right)^2\right] \quad .$$

In order to fix $\mathcal{M}'$, we remember that in the limit $\omega \to 0$ the system under consideration reduces to a system of free particles, that has a well known kernel $\mathcal{K}_{free}$ that we calculated in the previous subsection. But since

$$S_{osc}[x_{cl}, t_f, t_i] \xrightarrow{\omega\to 0} S_{free}[x_{cl}, t_f, t_i] \quad ,$$

it has to be

$$\mathcal{F}_{osc}(t_f - t_i) \xrightarrow{\omega\to 0} \mathcal{F}_{free}(t_f, t_i) \quad ;$$

[1] more precisely we will limit to say that the functional measure $\mathscr{D}[q(t)]$ is proportional to the measure of all Fourier coefficients, the proportionality factor being fixed later to satisfy the free particles limit $\omega \to 0$.

so being

$$\mathcal{F}_{osc}(t_f - t_i) \xrightarrow{\omega \to 0} \mathcal{M}' \quad ,$$

we can identify $\mathcal{M}' = \mathcal{F}_{free}(t_f, t_i)$, so finally we can write[1]

$$\mathcal{K}_{osc}(x_f, t_f; x_i, t_i) = \left(\frac{m\omega}{2\pi i \sin\big(\omega(t_f - t_i)\big)}\right)^{\frac{D}{2}} \exp\left\{ i \frac{m\omega}{2\sin\big(\omega(t_f - t_i)\big)} \Big((x_f^2 + x_i^2) \cdot \cos\big(\omega(t_f - t_i)\big) - 2x_f \cdot x_i \Big) \right\} . \tag{1.3.11}$$

1.3.3: *Perturbative Methods*

Unfortunately it is quite difficult to explicitly solve path integrals for more complicated systems, but we will see soon that we can develop a quite powerful tool to calculate perturbative expansions with path integrals introducing the so-called "correlation functions".

Heißenberg & Schrödinger pictures

We recall that there are two equivalent realizations of time evolution on a Hilbert Space $\mathscr{H}$: one is often referred to as Schrödinger picture, the other as Heißenberg picture. The former makes the states, *i.e.* the elements of Hilbert Space, carry any implicit time dependence (in fact they have to obey Schrödinger equation) while operators might have only explicit time dependence[2]; in the latter elements of $\mathscr{H}$ are constant in time, and operators carry all time dependence[3]. Now, since they must describe the same physical system, they have to be related by a unitary transformation, so $\mathscr{H}_S$ and $\mathscr{H}_H$ would be isomorphic. This is possible if time evolution is unitary, *i.e.* realized by a unitary operator $\mathcal{U}(t,s) \in L(\mathscr{H})$. As explained in Appendix B, this happens in most cases of physical interest. Furthermore, if time evolution is not unitary, the entire Schrödinger picture would be almost unmanageable, since in fact it consists in a mapping $t \to \mathscr{H}_t$, where $t \in \mathbb{R}$ is time, and $\mathscr{H}_t$ is the Space of states associated with the system at time t; now since the evolution is not unitary $\mathscr{H}_t$ and $\mathscr{H}_{t'}$ could be non-isomorphic, or worse even if $\mathscr{H}_t$ is a Hilbert Space, $\mathscr{H}_{t'}$ could not!

In fact in cases where we are not able to tell whether time evolution is unitary or not we must rely on Heißenberg picture, as in the case of interacting Quantum Field Theories, since in that picture the Hilbert Space describing the system is constant through time. If time evolution is unitary, we can use $\mathcal{U}(t,s)$ to associate $\mathscr{H}_H$ and $\mathscr{H}_S(t)$[4], with arbitrary $t \in \mathbb{R}$;

[1] being, as we have already seen, $\mathcal{K}_{osc}(x_f, t_f; x_i, t_i) \equiv \mathcal{K}_{osc}(x_f, (t_f - t_i); x_i, 0)$.

[2] in fact

$$\frac{\mathrm{d}A_S(t)}{\mathrm{d}t} = \frac{\partial A_S(t)}{\partial t} \quad ,$$

where A_S is an arbitrary operator in $L(\mathscr{H}_S)$.

[3] we recall that their time evolution is given by

$$\frac{\mathrm{d}A_H(t)}{\mathrm{d}t} = \left(\frac{\partial A_S(t)}{\partial t}\right)_H - i[A_H(t), \mathcal{H}_H(t)] \quad ,$$

where A_H is an arbitrary operator in $L(\mathscr{H}_H)$, and $\mathcal{H}_H$ is the Hamiltonian function.

[4] since time evolution is unitary, $\mathscr{H}_S(t') \equiv \mathcal{U}(t',t)\mathscr{H}_S(t)$ is isomorphic to $\mathscr{H}_S(t)$ for any $t', t \in \mathbb{R}$, so $\mathscr{H}_S$ has the same structure at any time.

in particular $\mathscr{H}_H \equiv \mathcal{U}^\dagger(t,t_0)\mathscr{H}_S(t)$, where t_0 is a fixed parameter. So $\mathscr{H}_S(t_0) \equiv \mathscr{H}_H$, while any other $\mathscr{H}_S(t)$ is isomorphic to it. $L(\mathscr{H}_H)$, the family of linear operators in Heißenberg picture is related to $L(\mathscr{H}_S)$ as usually done when we perform a unitary transformation, *i.e.* $L(\mathscr{H}_H) \equiv \mathcal{U}^\dagger(t,t_0)L(\mathscr{H}_S)\mathcal{U}(t,t_0)$.

So we immediately see that $\mathcal{K}(x_f,t_f;x_i,t_i) \equiv \langle x_f(t_f)|\mathcal{U}(t_f,t_i)|x_i(t_i)\rangle_S$ is the transition element between position eigenstates in Heißenberg picture:

$$\mathcal{K}(x_f,t_f;x_i,t_i) = \langle x_f(t_f)|\mathcal{U}(t_f,t_0)\mathcal{U}^\dagger(t_i,t_0)|x_i(t_i)\rangle_S = \langle x_f,t_f|x_i,t_i\rangle_H \ .$$

Now on we will use Heißenberg picture, omitting index $_H$, so $\mathcal{K}(x_f,t_f;x_i,t_i) = \langle x_f,t_f|x_i,t_i\rangle$. Up to this point, we would like to be able to calculate $\langle x_f,t_f|O(t_1)|x_i,t_i\rangle$, where $O(t_1) \in L(\mathscr{H})$ and $t_1 \in [t_i,t_f]$, with a path integral. This is easily done:

$$\begin{aligned}\langle x_f,t_f|O(t_1)|x_i,t_i\rangle &= \langle x_f(t_f)|\mathcal{U}(t_f,t_1)O(t_1)\mathcal{U}(t_1,t_i)|x_i(t_i)\rangle_S \\ &= \int \mathrm{d}x\mathrm{d}x'\, \mathcal{K}(x_f,t_f;x,t)\langle x(t)|O(t_1)|x'(t)\rangle_S\mathcal{K}(x',t;x_i,t_i) \quad ;\end{aligned}$$

if $O(t)_S$ is diagonal in the coordinates representation, *i.e.*

$$\langle x(t)|O(t_1)|x'(t)\rangle_S = O(x(t_1),t_1)\delta^D(x-x') \quad ,$$

then above expression reduces to a single path integral[1]:

$$\langle x_f,t_f|O(t_1)|x_i,t_i\rangle = \int\limits_{\substack{x(t_i)=x_i\\x(t_f)=x_f}} \mathscr{D}[x(t)]\ e^{iS[x,t_f,t_i]}O[x(t_1)] \quad . \tag{1.3.12}$$

This construction can be generalized to a time-ordered product of operators

$$\mathcal{T}\Big\{O_1(t_1)O_2(t_2)\cdots\Big\} \quad \text{with} \quad t_1,t_2,\ldots \in [t_i,t_f] \quad ,$$

calculation of transition elements with path integrals

obtaining, if the operators are all diagonal in the coordinates representation,

$$\langle x_f,t_f|\mathcal{T}\Big\{O_1(t_1)O_2(t_2)\cdots\Big\}|x_i,t_i\rangle = \int\limits_{\substack{x(t_i)=x_i\\x(t_f)=x_f}} \mathscr{D}[x(t)]\ e^{iS[x,t_f,t_i]}O_1[x(t_1)]O_2[x(t_2)]\cdots . \tag{1.3.13}$$

If $O_1(t_1) \equiv x^{i_1}(t_1)$, and so on, formula (1.3.13) reduces to:

$$\langle x_f,t_f|\mathcal{T}\Big\{x^{i_1}(t_1)x^{i_2}(t_2)\cdots\Big\}|x_i,t_i\rangle = \int\limits_{\substack{x(t_i)=x_i\\x(t_f)=x_f}} \mathscr{D}[x(t)]\ e^{iS[x,t_f,t_i]}x^{i_1}(t_1)x^{i_2}(t_2)\cdots ;$$

so we can define the *n-point propagator*, or *n-point correlation function*[2]

$$\langle\ x^{i_1}(t_1)\cdots x^{i_n}(t_n)\ \rangle \equiv \frac{\langle x_f,t_f|\mathcal{T}\Big\{x^{i_1}(t_1)\cdots x^{i_n}(t_n)\Big\}|x_i,t_i\rangle}{\langle x_f,t_f|x_i,t_i\rangle} \tag{1.3.14}$$

[1] to make this possible it is necessary that $t \in [t_i,t_f]$, in order to identify the integration over x as one of the slices of the path at an intermediate time t.

[2] with this definition the zero point correlation function, *i.e.* $\langle\ 1\ \rangle = 1$; but we will use a different notation here, we define $\langle\ 1\ \rangle \equiv \langle x_f,t_f|x_i,t_i\rangle$: so $\langle x_f,t_f|\mathcal{T}\Big\{x^{i_1}(t_1)\cdots x^{i_n}(t_n)\Big\}|x_i,t_i\rangle = \langle\ 1\ \rangle\langle\ x^{i_1}(t_1)\cdots x^{i_n}(t_n)\ \rangle$.

through a path integral. We will see in a moment that there is a beautiful way to calculate such correlations functions with n arbitrary knowing only the 2-point propagator. Define a "generating functional"

Generating Functional

$$Z[J] = \int\limits_{\substack{x(t_i)=x_i \\ x(t_f)=x_f}} \mathscr{D}[x(t)]\; e^{i\left(S[x,t_f,t_i]+\int \mathrm{d}t\, J_i(t)x^i(t)\right)} \quad , \tag{1.3.15}$$

where the "sources" $J(t)$ are D-dimensional vectors. It is immediate to see that the n-point propagator can be written in terms of functional derivatives of generating functional[1]:

$$\langle\, x^{i_1}(t_1)\cdots x^{i_n}(t_n)\,\rangle = \frac{1}{Z[J]}(-i)^n \frac{\delta}{\delta J_{i_1}(t_1)}\cdots\frac{\delta}{\delta J_{i_n}(t_n)} Z[J]\Big|_{J=0} \quad . \tag{1.3.16}$$

Now if $S[x,t_f,t_i]$ reduces to $S_2[x,t_f,t_i] = -\int_{t_i}^{t_f}\int_{t_i}^{t_f} \mathrm{d}t\mathrm{d}t'\, \frac{1}{2}K_{ij}(t,t')x^i(t)x^j(t')$, where $K_{ij}(t,t')$ is a differential operator, $Z[J]$ reduces to a Gaussian Integral[2,3]

$$Z[J] = \int\limits_{\substack{x(t_i)=x_i \\ x(t_f)=x_f}} \mathscr{D}[x(t)]\; e^{-i\left(\frac{1}{2}K_{ij}x^ix^j - J_ix^i\right)} = \mathrm{Det}^{-\frac{1}{2}}\left[\frac{iK}{2\pi}\right]\mathrm{Det}^{-\frac{1}{2}}\left[\frac{i\mathscr{A}}{2\pi}\right]\; e^{\frac{i}{2}(K^{-1})^{ij}J_iJ_j} \quad .$$

So we immediately deduce by derivation the so called "Wick Theorem", *i.e.* that any odd-point propagator must vanish, and any even-point propagator is calculated by means of all possible products of 2-point propagators[4]:

$$\begin{aligned}
\langle\, x^i \,\rangle_2 &= 0 \quad , \\
\langle\, x^{i_1}x^{i_2} \,\rangle_2 &= -i(K^{-1})^{ij} \quad , \\
\langle\, x^{i_1}\cdots x^{i_{n+1}} \,\rangle_2 &= 0 \quad , \qquad \text{(Wick Theorem)} \\
\langle\, x^{i_1}\cdots x^{i_{2n}} \,\rangle_2 &= \sum_{\substack{\text{permutations of} \\ i_1\ldots i_{2n}}} \left(-i(K^{-1})^{i_{p1}i_{p2}}\right)\cdots\left(-i(K^{-1})^{i_{p(2n-1)}i_{p2n}}\right) \quad .
\end{aligned}$$

Unfortunately only when we deal with quadratic Lagrangian functions we can write the action as $S_2[q,t_f,t_i] = -\int_{t_i}^{t_f} \mathrm{d}t\, \frac{1}{2}K_{ij}(t)q^i(t)q^j(t)$, following the method described at the beginning of Section 1.3. Nevertheless we can use these tools in the case of a general action $S[x,t_f,t_i]$ if we are able to write it as $S_2[x,t_f,t_i] + S_{int}[x,t_f,t_i]$: in fact $\mathcal{K}(x_f,t_f;x_i,t_i)$ thus

[1] the functional derivative is defined (in the unidimensional case, the generalization is straightforward) as the following: given a functional $F[f(x)]$ its functional derivative is the fraction

$$\frac{\delta F[f(x)]}{\delta f(y)} \equiv \lim_{\epsilon\to 0}\frac{F[f(x)+\epsilon\delta(x-y)]-F[f(x)]}{\epsilon} \quad .$$

[2] using the usual compact multi-index notation.

[3] see Appendix C and Section 1.2.3 for an explanation of the factor $\mathrm{Det}^{-1/2}\left[\frac{i\mathscr{A}}{2\pi}\right]$.

[4] we have used $_2$ to make explicit that this Theorem holds if the averages are calculated with an action S_2. Since now on we will always calculate averages of that kind, the index $_2$ will be omitted.

becomes

$$\mathcal{K}(x_f,t_f;x_i,t_i) = \int\limits_{\substack{x(t_i)=x_i\\ x(t_f)=x_f}} \mathscr{D}[x(t)]\; e^{iS_2[x,t_f,t_i]} e^{iS_{int}[x,t_f,t_i]} = \langle\, 1\,\rangle \langle\, e^{iS_{int}[x,t_f,t_i]}\,\rangle \quad ;$$

and since the exponential can be expanded in a power series, and the average of a sum is the sum of the averages we obtain:

$$\mathcal{K}(x_f,t_f;x_i,t_i) = \langle\, 1\,\rangle \Big(1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \langle\, S_{int}^n\,\rangle\Big) \quad .$$

If this series depends on a small parameter, this expression could be seen as the perturbative expansion we sought[1]. We stress that with this construction all we need to know to perform perturbative calculations is the 2-point propagator, *i.e.* the inverse of the kinetic operator $K_{ij}(t,t')$ appearing in $S_2[x,t_f,t_i]$.

1.3.4: *Some Perturbative Calculations*

Let's see how to apply the powerful tools of the previous section to some simple case. In fact we will use generating functionals, propagators and perturbative expansions almost everywhere throughout this work, but some simple example will help us get acquainted with such techniques.

Let's start with a well known problem, a system of uncoupled harmonic oscillators: we will calculate the 2-point propagator for such a theory (recall that S_{osc} is a S_2-type action), in a particular case. It is particularly useful to analyze this propagator even if we already solved the path integral for such a system, because a quite crucial problem will emerge quite naturally in performing that calculation, related with the impossibility to define coherently path integrals in real time without recurring to analytic continuation of calculations in imaginary time. We already encountered a first problem: path integrals in real time involve "imaginary" Gaussian integrals, like $\int \mathscr{D}[p] e^{ip^2/2}$, also known as Fresnel integrals, that are analytic continuations of well-known real Gaussian integrals, like $\int \mathscr{D}[p] e^{-p^2/2}$, see Appendix C for further details. This underlies a deeper problem: we are not able to define a functional measure in the space of paths in real time due to oscillatory factors e^{iS}, but we can do that if we continue time analytically to purely imaginary values; this procedure is called "Wick Rotation", and consists in a redefinition of time $t \to -it$: with that procedure the oscillatory factor becomes a real exponential e^{-S_E}, where S_E is the so-called Euclidean action, and we can define Wiener measures in path space in such a case. So if we want to be more rigorous and solve ambiguities, we have to perform calculations in Euclidean (imaginary) time, and then "rotate back" to real time through analytic continuation.

Euclidean time & Wick rotation

Returning to the propagator for a theory of harmonic oscillators, we will see that in real time we have poles on the integration path, and we have an ambiguity on how to "pass by"

[1] recall that such a parameter could be $\hbar$ that we set to 1 throughout our derivation but else would have appeared as a factor $\hbar^{-n}$ in the perturbative expansion; also a suitable choice of $t_f - t_i$ could serve as the small parameter as it appears in the power series (only if the Hamiltonian is time-independent).

them, but in Euclidean time there are no poles on the integration path so the ambiguity is solved, and analytic continuation of the result gives us back the real time propagator. We are not able to perform calculations in the case of an arbitrary $\mathcal{K}_{osc}(x_f, t_f; x_i, t_i)$[1], so we will restrict to the calculation of $\mathcal{Z}(t_f, t_i) \equiv \int \mathrm{d}x \, \langle x, t_f | x, t_i \rangle$: $\langle x, t_f | x, t_i \rangle$ is a path integral, with a periodic boundary condition $x(t_f) = x(t_i) = x$, and such boundary value is then integrated over all its possible values; this integration over the boundary value is equivalent to modify the space of paths we are considering in defining the functional measure: we pass from integration over all possible paths with fixed endpoints to integration over all possible paths with the initial point equivalent to the final one, and we will call this constraint "**P**eriodic **B**oundary **C**onditions", or *PBC*. So we can write

$$\mathcal{Z}(t_f, t_i) = \int_{PBC} \mathscr{D}[x(t)] \; e^{iS[x, t_f, t_i]} \quad . \tag{1.3.17}$$

It is straightforward that all the considerations about generating functionals, propagators *etc.* we did for $\mathcal{K}_{osc}(x_f, t_f; x_i, t_i)$ remain valid for $\mathcal{Z}(t_f, t_i)$: it is sufficient to substitute in any formula fixed endpoints conditions with *PBC*. We will see the usefulness of $\mathcal{Z}$ immediately as we will consider the Euclidean case. By now we will limit ourselves to find the 2-point propagator for $\mathcal{Z}$ in the case of Harmonic Lagrangian function. First of all since time translation is a symmetry of Harmonic systems, we can write:

$$\mathcal{Z}(t_f, t_i) \equiv \mathcal{Z}(\beta) = \int_{PBC} \mathscr{D}[x(t)] \; e^{iS[x, \frac{\beta}{2}, -\frac{\beta}{2}]} \quad \text{with} \quad \beta \equiv t_f - t_i \quad .$$

Then we note that periodic boundary conditions allow us to integrate by parts in the action, since the boundary term vanishes, so we can write Harmonic action, using multi-index notation as usual:

$$S_{osc} = \frac{1}{2} K_{ij} x^i x^j \quad ,$$

with $K_{ij}(t, t') = m(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2)\delta(t - t')\delta_{ij}$. So in order to find the propagator all we have to do is find the inverse of $K_{ij}(t, t')$; we have to solve the distributional equation

$$\int \mathrm{d}t'' \, K_{ij}(t, t'')(K^{-1})^{jk}(t'', t') = \delta_i{}^k \delta(t - t') \quad ,$$

i.e.:

$$\int \mathrm{d}t'' \, m\delta_{ij}\delta^D(t - t'')(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2)(K^{-1})^{jk}(t'', t') = m\delta_{ij}(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2)(K^{-1})^{jk}(t, t') = \delta_i{}^k \delta(t - t') \, ,$$

$$\Rightarrow (K^{-1})^{jk}(t, t') = \frac{\delta^{jk}}{m} K^{-1}(t - t') \quad ,$$

[1] because even if the Lagrangian function of such a system is a quadratic function, we are not able to find a solution for the inverse of the Harmonic kinetic operator that satisfies either general boundary conditions or vanishing ones, as we would have to if we wanted to solve the general case. We will see that for a free system we know a solution of the inverse equation that satisfies vanishing boundary conditions, so we are able to find the propagator for a general Kernel.

here we guess that since $K(t,t') \equiv K(t-t')$ the same property holds for $K^{-1}(t,t')$; if we can find a solution we prove that guess is right, since the solution is unique. So we have to seek a solution of:

$$(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2)K^{-1}(x) = \delta(x) \quad ,$$

so if we consider the limit $\beta \to \infty$ we can apply Fourier transform, so the equation for $\hat{K}^{-1}(k)$ reads

$$\begin{aligned} &-(k^2-\omega^2)\hat{K}^{-1}(k) = 1 \quad , \\ &\Rightarrow \hat{K}^{-1}(k) = -\frac{1}{k^2-\omega^2} \quad , \\ &\Rightarrow K^{-1}(x) = -\int \frac{\mathrm{d}k}{2\pi}\frac{1}{k^2-\omega^2}e^{ikx} \quad ; \end{aligned}$$

here we see that there are poles on the integration path, as we anticipated, and we don't know what is the right prescription to manage these singularities. We are thus forced to perform calculations in Euclidean time and recover the real time result through analytic continuation. But what is the difference between real and Euclidean time calculations? Since $S[x,t_f,t_i] = \int_{t_i}^{t_f} \mathrm{d}t\, \mathscr{L}(x(t),\dot{x}(t),t)$ we have, performing the transformation $t' = it$:

$$iS[x,t_f,t_i] = i\int_{t_i'}^{t_f'} \mathrm{d}(-it')\, \mathscr{L}(x(t'),i\dot{x}(t'),-it') = -S_E[x,t_f,t_i]$$

where $S_E[x,t_f,t_i] = -\int_{t_i}^{t_f} \mathrm{d}t\, \mathscr{L}(x(t),i\dot{x}(t),-it)$, recalling $t' = t$. So Euclidean time path integral becomes

$$\mathcal{K}(x_f,t_f;x_i,t_i) = \int_{\substack{x(t_i)=x_i \\ x(t_f)=x_f}} \mathscr{D}[x(t)]\; e^{-S_E[x,t_f,t_i]} \quad , \tag{1.3.18}$$

$$\mathcal{Z}(t_f,t_i) = \int_{PBC} \mathscr{D}[x(t)]\; e^{-S_E[x,t_f,t_i]} \quad ; \tag{1.3.19}$$

and it is straightforward to generalize generating functionals, correlation functions and

Wick Theorem[1] in Euclidean time:

$$Z[J] = \int_{\text{boundaries}} \mathscr{D}[x(t)]\, e^{-\left(S_E[x,t_f,t_i] - \int \mathrm{d}t\, J_i(t) x^i(t)\right)} \quad , \tag{1.3.20}$$

$$\langle\, x^{i_1}(t_1) \cdots x^{i_n}(t_n)\, \rangle = \frac{1}{Z[J]} \frac{\delta}{\delta J_{i_1}(t_1)} \cdots \frac{\delta}{\delta J_{i_n}(t_n)} Z[J]\Big|_{J=0} \quad , \tag{1.3.21}$$

$$\begin{aligned} \langle\, x^{i_1} x^{i_2}\, \rangle_2 &= (K^{-1})^{ij} \quad , \\ \langle\, x^{i_1} \cdots x^{i_{n+1}}\, \rangle_2 &= 0 \quad , \\ \langle\, x^{i_1} \cdots x^{i_{2n}}\, \rangle_2 &= \sum_{\substack{\text{permutations of} \\ i_1 \ldots i_{2n}}} (K^{-1})^{i_{p1} i_{p2}} \cdots (K^{-1})^{i_{p(2n-1)} i_{p2n}} \quad . \end{aligned} \tag{Eucl. Wick Th.}$$

In the case of Harmonic oscillators Euclidean action becomes:

$$S_{E,osc}[x, t_f - t_i] = \frac{m}{2} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} \mathrm{d}t\, (\dot{x}(t)^2 + \omega^2 x(t)^2) \quad ,$$

so integrating by parts[2] we obtain the usual form $S_2 = \frac{1}{2} K_{ij} x^i x^j$ with $K_{ij}(t,t') = m(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2)\delta(t-t')\delta_{ij}$. Now repeating the same procedure we did in real time for the propagator of $\mathcal{Z}(\beta)$ we find that

$$(K^{-1})^{jk}(t,t') = \frac{\delta^{jk}}{m} K^{-1}(t-t')$$

with

$$K^{-1}(x) = \int \frac{\mathrm{d}k}{2\pi} \frac{1}{k^2 + \omega^2} e^{ikx} \quad ;$$

it's clear that we don't have any pole in the integration path, so a usual integration in the complex plane gives us

$$(K^{-1})^{ij}(t,t') = \frac{\delta^{ij}}{2m\omega} e^{-\omega|t-t'|} \quad .$$

Clearly the Euclidean propagator is

$$\langle\, x^i(t) x^j(t')\, \rangle_E = \frac{\delta^{ij}}{2m\omega} e^{-\omega|t-t'|} \quad ,$$

the real time propagator is given by analytic continuation of the Euclidean one[3]:

$$\langle\, x^i(t) x^j(t')\, \rangle_R = \frac{\delta^{ij}}{2m\omega} e^{-i\omega|t-t'|} \quad .$$

[1] we recall that in Euclidean time we have $S_{E2} = \int \mathrm{d}t\mathrm{d}t'\, \frac{1}{2} K_{ij}(t,t') x^i(t) x^j(t')$.

[2] in the calculation of $\mathcal{Z}$, where the integration by parts is allowed without boundary terms.

[3] in that case absolute value is used to write the propagator in a compact way, but when making analytic continuation we have to consider separately the cases $t - t' < 0$ and $t - t' > 0$, and then restore the compact notation.

We note that the result obtained through analytic continuation is the same we would have obtained with the so-called Feynman prescription

$$K^{-1}(x) = -\int \frac{\mathrm{d}k}{2\pi}\frac{1}{k^2-\omega^2+i\epsilon}e^{ikx}$$

for the treatment of poles on integration path. Let's see now the physical meaning of $\mathcal{Z}(\beta)$ in Euclidean time: we have seen in Eq. (1.3.19) that it can be written in a path integral form, but if we remember its own definition[1] and perform Wick rotation we obtain:

$$\mathcal{Z}(\beta) \equiv \int \mathrm{d}x\, \langle x|e^{-\beta\mathcal{H}}|x\rangle = \mathrm{Tr}\, e^{-\beta\mathcal{H}} \quad ;$$

it is clear now that $\mathcal{Z}(\beta)$ is the *partition function* of a statistical system with an Hamiltonian $\mathcal{H}$ at temperature $T = (k\beta)^{-1}$, where k is the Boltzmann constant.

So, at least in the asymptotic limit $\beta \to \infty$, we know the 2-point propagator for the partition function of Harmonic systems, and also $\langle\ 1\ \rangle$: calculating $\int \mathrm{d}x\, \mathcal{K}_{osc}(x, \frac{\beta}{2}; x, -\frac{\beta}{2})$, we obtain (performing Wick rotation in Eq. (1.3.11) and then integrating),

$$\langle\ 1\ \rangle = \left(\frac{1}{2(\cosh(\omega\beta)-1)}\right)^{\frac{D}{2}} \quad .$$

We have thus all we need to make perturbative calculations for a system with action $S_E = S_{osc,E} + S_{int,E}$: consider for example a system of Harmonic oscillators subjected to a perturbation potential

$$V_{int}(x) = g_{ijk}x^i x^j x^k + \lambda_{ijkl}x^i x^j x^k x^l \quad ;$$

we can see the effect of that perturbation on the partition function for large β by means of a perturbative series in g and λ: in that case $S_{int,E} = \int \mathrm{d}t\, (g_{ijk}x^i x^j x^k + \lambda_{ijkl}x^i x^j x^k x^l)$, so we have[2]

$$\mathcal{Z}_{int}(\beta) = \langle\ 1\ \rangle\Big(1 - \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} \mathrm{d}t\, \left(g_{ijk}\langle\ x^i(t)x^j(t)x^k(t)\ \rangle + \lambda_{ijkl}\langle\ x^i(t)x^j(t)x^k(t)x^l(t)\ \rangle\right)\Big) + O(g^2+\lambda^2)$$

but since odd-point propagators vanish we see that the only first-order contribution to perturbation is given by the quartic potential, in particular recalling that $\langle\ x^i(t)x^j(t)\ \rangle_E = \delta^{ij}/2m\omega$, that λ_{ijkl} has to be completely symmetric in the exchange of two indices (the eventual antisymmetric part would vanish when contracted with four xs) and that $\langle\ 1\ \rangle \equiv \mathcal{Z}_{osc}(\beta)$, we obtain:

$$\mathcal{Z}_{int}(\beta) = \mathcal{Z}_{osc}(\beta)\Big(1 - \frac{3\beta\lambda_{i\ k}^{\ i\ k}}{4m^2\omega^2}\Big) + O(g^2+\lambda^2) = \mathcal{Z}_{osc}(\beta)\, e^{-\beta\frac{3\lambda_{i\ k}^{\ i\ k}}{4m^2\omega^2}} + O(g^2+\lambda^2) \quad ,$$

[1] in the case of a time-independent Hamiltonian, *i.e.* when time translation is a symmetry of the system and $\mathcal{Z}(t_f, t_i) \equiv \mathcal{Z}(t_f - t_i, 0)$.

[2] remembering that in Euclidean time:

$$\int_{\text{boundaries}} \mathscr{D}[x(t)]\, e^{-\left(S_{E2}+S_{int,E}\right)} = \langle\ 1\ \rangle\langle\ e^{-S_{int,E}}\ \rangle = \langle\ 1\ \rangle\Big(1 + \sum_{n=1}^{\infty}\frac{(-)^n}{n!}\langle\ S_{int,E}^n\ \rangle\Big) \quad .$$

with $\lambda_{i\ k}^{\ i\ k} \equiv \lambda_{ijkl}\delta^{ij}\delta^{kl}$.

A useful application of this result is to calculate the first order effect of the anharmonic perturbation on the ground-state energy of the system. In fact if we calculate the trace in the partition function using a basis of Hamiltonian eigenfunctions we obtain:

$$\mathcal{Z}(\beta) = \sum_n \langle n|e^{-\beta\mathcal{H}}|n\rangle = \sum_n e^{-\beta E_n} = e^{-\beta E_0}\left(1 + \sum_n e^{-\beta(E_n - E_0)}\right) \overset{\beta\to\infty}{\sim} e^{-\beta E_0} \quad ;$$

so the ground-state energy shift is:

$$\Delta E_0 = E_0^{int} - E_0^{osc} = \lim_{\beta\to\infty} -\frac{1}{\beta}\ln(Z_{int}/Z_{osc}) = \frac{3}{4}\frac{\lambda_{i\ k}^{\ i\ k}}{m^2\omega^2} \quad .$$

As a last simple example we will find propagators for a free theory, and use them to make perturbative calculations in the case of a system of particles with the same mass subjected to an arbitrary external potential (using evolution time as the small parameter). In order to find propagators we will use directly Euclidean time, and in that case the action reads $S_{free,E}[x,\beta] = \int_0^\beta \mathrm{d}t\, \frac{m}{2}\dot{x}(t)^2$, with $\beta \equiv (t_f - t_i)$ and recalling that since free Lagrangian does not depend explicitly on time, time translation is a symmetry of the system so $\mathcal{K}(x_f,t_f,x_i,t_i) \equiv \mathcal{K}(x_f,\beta;x_i,0)$. Since the free Lagrangian is also a quadratic function, we can do the usual x_{cl}-q split, remembering that in the Euclidean case we obtain:

$$\mathcal{K}(x_f,t_f;x_i,t_i) = e^{-S[x_{cl,E},t_f,t_i]} \int\limits_{VBC} \mathscr{D}[q(t)]\, e^{-S_{E2}[q,t_f,t_i]} \quad ; \tag{1.3.22}$$

that reads in this case:

$$S[x_{cl,E},\beta] = \frac{m}{2}\frac{(x_f - x_i)^2}{\beta} \quad ,$$
$$S_{E2}[q,\beta] = \int_0^\beta \mathrm{d}t\mathrm{d}t'\, \frac{1}{2}m\delta_{ij}(-\frac{\mathrm{d}^2}{\mathrm{d}t^2})\delta(t-t')q^i(t)q^j(t') \quad .$$

So we recognize again the usual form $S_2 = \frac{1}{2}K_{ij}q^iq^j$. But now in order to find $(K^{-1})^{ij}$ we will proceed in a slightly different way: we perform the mode expansion we did in Section 1.3.2, so we can perform the integration over t in S_{E2}, obtaining a form[1]

$$S_2 = \lim_{N,P\to\infty} \frac{1}{2}K_{ij}C^iC^j \quad ,$$

where K_{ij} depends only on discrete indices (it is a real matrix), and so it is much easier to invert.

If one defines the Fourier expansion:

$$q(t) = \lim_{N\to\infty} \sum_{n=1}^{N} \sqrt{\frac{2}{\beta}}\, C_n \sin\left(\frac{n\pi t}{\beta}\right) \quad \text{where} \quad C_n \in \mathbb{R}^D \quad , \tag{1.3.23}$$

[1] $C^i \equiv C_n^i$ are Fourier coefficients of expansion, recalling notation of Section 1.3.2.

the integration over t in $S_{E2}[q,\beta]$ is straightforward, obtaining the usual form[1]

$$S_2 = \lim_{N,P\to\infty} \frac{1}{2} K_{ij} C^i C^j$$

with $C \in \mathbb{R}^{DN}$ and

$$K_{ij}^{np} = \frac{m\pi^2 n^2}{\beta^2}\delta_{ij}\delta^{np} \quad ,$$

so the inverse is immediately found as:

$$(K^{-1})_{np}^{ij} = \frac{\beta^2}{m\pi^2 n^2}\delta^{ij}\delta_{np} \quad ,$$

and since integration over all possible paths vanishing at the boundaries is equivalent to an integration over all Fourier coefficients C_n^i, we can find propagators of the Cs in a standard way obtaining:

$$\langle\, C_n^i C_p^j \,\rangle_E = \frac{\beta^2}{m\pi^2 n^2}\delta^{ij}\delta_{np} \quad ,$$

so using Eq. (1.3.23) we find

$$\langle\, q^i(t) q^j(t') \,\rangle_E = \lim_{N,P\to\infty} \sum_{n=1}^{N}\sum_{p=1}^{P} \frac{2}{\beta}\sin\Big(\frac{\pi n t}{\beta}\Big)\sin\Big(\frac{\pi p t'}{\beta}\Big)\langle\, C_n^i C_p^j \,\rangle_E = \frac{\delta^{ij}}{m}\Delta(t,t')\,, \quad (1.3.24)$$

with

$$\Delta(t,s) = \sum_{n=1}^{\infty}\Big[\frac{2\beta}{\pi^2 n^2}\sin\Big(\frac{\pi n t}{\beta}\Big)\sin\Big(\frac{\pi n s}{\beta}\Big)\Big] \quad . \quad (1.3.25)$$

In order to perform calculations, we need to know the distributional meaning of $\Delta(t,s)$ in the space of continuous functions on $[0,\beta]$ vanishing at the boundaries, *i.e.* the algebra of continuous functions with compact support $\kappa([0,\beta])$. Since the propagator (1.3.24) has to be the inverse of the kinetic operator $m\delta_{ij}\delta(t-t')(-\frac{\mathrm{d}^2}{\mathrm{d}t^2})$ we expect that $\Delta(t,s) \in \kappa([0,\beta])^*$ [2] satisfies the equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Delta(t,s) = \delta(t-s) \quad . \quad (1.3.26)$$

Let's see directly if it is the case: first of all we note that $\Delta(t,s)$ is symmetric in t and s, so if Eq. (1.3.26) holds, it holds even

$$-\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Delta(t,s) = \delta(t-s) \quad ;$$

denoting a derivation of Δ with a $^\bullet$, to the left of Δ if we derive with respect to the first argument, to the right if we derive with respect to the second, we have:

$$-{}^{\bullet\bullet}\Delta(t,s) = -\Delta^{\bullet\bullet}(t,s) = \sum_{n=1}^{\infty}\frac{2}{\beta}\sin\Big(\frac{\pi n t}{\beta}\Big)\sin\Big(\frac{\pi n s}{\beta}\Big) \quad , \quad (1.3.27)$$

[1] the cutoff N in the Fourier expansion is used in order to make K a finite $DN \times DN$ dimensional matrix.
[2] $\kappa([0,\beta])^*$ is the dual space of $\kappa([0,\beta])$.

is it $\delta(t-s)$? Yes, indeed: the definition of $\delta(t-s)$ as a functional in $\kappa([0,\beta])^*$ is

$$\delta(t-s)[f(t)] \equiv \int_0^\beta \mathrm{d}t\, \delta(t-s) f(t) = f(s) \quad \text{with} \quad f(t) \in \kappa([0,\beta]) \quad ;$$

so if we search the coefficients $D_n(s)$ of a Fourier expansion

$$\delta(t-s) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\beta}} D_n(s) \sin\Big(\frac{\pi n t}{\beta}\Big)$$

we have to solve[1]

$$\sum_{n,p} \frac{2}{\beta} D_n(s) F_p \int_0^\beta \mathrm{d}t\, \sin\Big(\frac{\pi n t}{\beta}\Big) \sin\Big(\frac{\pi p t}{\beta}\Big) = \sum_{n,p} D_n(s) F_p \delta_{np}$$

$$\Rightarrow D_n(s) = \sqrt{\frac{2}{\beta}} \sin\Big(\frac{\pi n s}{\beta}\Big) \quad ,$$

$$\Rightarrow \delta(t-s) = \sum_{n=1}^{\infty} \frac{2}{\beta} \sin\Big(\frac{\pi n t}{\beta}\Big) \sin\Big(\frac{\pi n s}{\beta}\Big) = -\overset{\bullet\bullet}{\Delta}(t,s) \quad .$$

Analysis of Distributions in $\kappa([0,\beta])^*$ by means of Fourier series

So a general solution of (1.3.26), symmetric in t and s and in $\kappa([0,\beta])^*$ is:

$$\Delta(t,s) = \frac{1}{\beta}[(\beta - t)s\theta(t-s) + (\beta - s)t\theta(s-t)] \quad ,$$

where $\theta(t-s) \in \kappa([0,\beta])^*$ is defined as

$$\theta(t-s)[f(t)] \equiv \int_0^\beta \mathrm{d}t\, \theta(t-s) f(t) = \int_s^\beta \mathrm{d}t\, f(t) \quad \text{with} \quad f(t) \in \kappa([0,\beta]) \quad .$$

From its own definition $\theta(t-s)$ has a Fourier expansion

$$\theta(t-s) = 2\Big(\sum_{n=1}^{\infty} \frac{1}{\pi 2n}\Big[\cos\Big(\frac{\pi 2ns}{\beta}\Big) - 1\Big] \sin\Big(\frac{\pi 2nt}{\beta}\Big) + \sum_{n=0}^{\infty} \frac{1}{\pi(2n+1)}\Big[\cos\Big(\frac{\pi(2n+1)s}{\beta}\Big) + 1\Big] \sin\Big(\frac{\pi(2n+1)t}{\beta}\Big)\Big) \quad ;$$

so it is directly checked that $\theta(0) \equiv \theta(t-t)$ satisfies for $f(t) \in \kappa([0,\beta])$:

$$\theta(0)[f(t)] \equiv \int_0^\beta \mathrm{d}t\, \theta(0) f(t) = \frac{1}{2}\int_0^\beta \mathrm{d}t\, f(t) \equiv \frac{1}{2}[f(t)] \quad ,$$

where $1/2$ is an element of $\kappa([0,\beta])^*$, so we can say that in $\kappa([0,\beta])^*$, $\theta(0) = 1/2$; therefore we obtain that

$$\Delta(t,t) = t - \frac{1}{\beta}t^2 \tag{1.3.28}$$

as an element of $\kappa([0,\beta])^*$. Now that we know the behavior of Δ in time integrals we can perform perturbative calculations for a wide class of physical systems. In fact in most cases

[1] expanding $f(t)$ in a Fourier series with coefficients F_n.

of physical interest the quantum Hamiltonian function of the system is $\mathcal{H} = -\frac{m}{2}\nabla^2 + V(x)$, where ∇^2 is the Laplace operator in D dimensions and $V(x) \in C_\infty(\mathbb{R}^D)$, the infinitely differentiable functions in $\mathbb{R}^D$. The Euclidean action corresponding to such a system is simply:

$$S_E(x, t_f, t_i) = S_{free,E}(x, \beta) + \int_0^\beta \mathrm{d}t\, V\big(x(t)\big) \quad ,$$

where as usual $\beta = t_f - t_i$; so we are able to write

$$\mathcal{K}(x_f, t_f; x_i, t_i) = \langle\ 1\ \rangle \Big(1 + \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \int_0^\beta \mathrm{d}t_1 \cdots \mathrm{d}t_n \ \langle\ V\big(q(t_1) + x_{cl}(t_1)\big) \cdots V\big(q(t_n) + x_{cl}(t_n)\big)\ \rangle\Big)$$

with $\langle\ q^i(t) q^j(t')\ \rangle = (\delta^{ij}/m)\Delta(t, t')$ and $x_{cl}(t) = [(x_f - x_i)/\beta]t + x_i$. This is not yet a perturbative series, because we don't have defined a controlling parameter that permits to approximate the series when it is small. If we use β as a perturbative parameter we are interested to write a series

$$\mathcal{K}(x_f, t_f; x_i, t_i) = a_0(x_f, x_i) + a_1(x_f, x_i)\beta + a_2(x_f, x_i)\beta^2 + \dots \quad ;$$

we will not perform all the boring calculations here, we will limit to describe the methodology: first of all we expand the potential $V(x)$ in a Taylor series around the initial point x_i, obtaining a series[1]

$$V(x) = V(x_i) + (q^i(t) + \frac{\xi^i}{\beta}t)\partial_i V(x_i) + \frac{1}{2}(q^i(t) + \frac{\xi^i}{\beta}t)(q^j(t) + \frac{\xi^j}{\beta}t)\partial_i\partial_j V(x_i) + \dots \quad ,$$

then we perform averages using the 2-point propagator $\langle\ q^i(t) q^j(t')\ \rangle$ and finally solve integrals remembering the distributional meaning of $\Delta(t, s)$; for example one has to solve the integral

$$\int_0^\beta \mathrm{d}t\, \Delta(t, t) = \int_0^\beta \mathrm{d}t\, \Big(t - \frac{1}{\beta}t^2\Big) = \frac{1}{6}\beta^2 \quad .$$

As a mention we write here the values of the first three coefficients if $x_f = x_i = x$, see[14] for further details:

$$a_0(x, x) = \langle\ 1\ \rangle = \Big(\frac{m}{2\pi\beta}\Big)^{\frac{D}{2}} \quad ,$$

$$a_1(x, x) = -\Big(\frac{m}{2\pi\beta}\Big)^{\frac{D}{2}} V(x) \quad ,$$

$$a_2(x, x) = \Big(\frac{m}{2\pi\beta}\Big)^{\frac{D}{2}} \Big(\frac{1}{2}V(x)^2 - \frac{1}{12m}\nabla^2 V(x)\Big) \quad .$$

[1] remembering that

$$x(t) = x_{cl}(t) + q(t) = q(t) + \frac{x_f - x_i}{\beta}t + x_i \quad ;$$

and defining $\xi = x_f - x_i$.

Chapter 2

Quantum Fields or Quantum Particles?

"The vacuum current of a charged Dirac field [...] can be related to the dynamical properties of a 'particle' with space-time coordinates that depend upon a proper-time parameter."

Julian Schwinger

Previous chapter analyzed the usefulness of functional integration describing the time evolution of a system of Quantum mechanical particles. We will see in this chapter that path integrals can be extended quite easily to Quantum Field Theories and provide an alternative method to calculate propagators and derive Feynman rules in such an environment. Then we see that using the Schwinger proper-time representation we can perform Quantum Field Theory calculations[1] through the path integral of a Quantum mechanical particle moving in space-time[2].

2.1: QFT Path Integrals

In order to construct a functional integral formulation of Quantum Field Theory we need to have a deeper insight on the mathematical framework describing a relativistic system, the so-called "second quantization", so we will introduce it briefly.

[1] in particular the calculation of Effective Actions.
[2] these methods are often called "world-line path integral representations" for Effective Actions.

2.1.1: *Fock Spaces*

The Hilbert space in the Quantum mechanical description of a single particle depends on the system we are considering: as an example a Schrödinger particle of spin one half has $\mathscr{H} = L^2(\mathbb{R}^3, dx; \mathbb{C}^2) \equiv L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, where $\equiv$ means natural isomorphism; *i.e.* the set of pairs $\{\psi_1(x), \psi_2(x)\}$ of square-integrable functions (dx is the Lebesgue measure). If we want to deal with a relativistic system, in which the number of particles is not fixed, we need to define a different framework, called Fock space[15]:

Definition 2.1.1 (Fock Space over $\mathscr{H}$)**.** Let $\mathscr{H}$ be a Hilbert space and $\mathscr{H}^n = \mathscr{H} \otimes \cdots \otimes \mathscr{H}$ an n-fold tensor product. Setting $\mathscr{H}^0 = \mathbb{C}$ we define the Fock Space over $\mathscr{H}$ as:

$$\mathscr{F}(\mathscr{H}) = \bigoplus_{n=0}^{\infty} \mathscr{H}^n \quad .$$

However $\mathscr{F}(\mathscr{H})$ is too large, since quantum particles are indistinguishable, so we have to restrict to two of its subspaces, the symmetric and antisymmetric Fock spaces. Let's see how to construct such subspaces: let $\mathscr{P}_n$ be the permutation group on n elements and $\{\phi_k\}$ be a basis for $\mathscr{H}$. For each $\sigma \in \mathscr{P}_n$, we define an operator (also denoted by σ) on basis elements of $\mathscr{H}^n$ by

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$$\sigma(\phi_{k_1} \otimes \cdots \otimes \phi_{k_n}) = \phi_{k_{\sigma(1)}} \otimes \cdots \otimes \phi_{k_{\sigma(n)}} \quad ;$$

σ extends by linearity to a bounded operator of norm one on $\mathscr{H}^n$ so we can define $S_n = (1/n!) \sum_{\sigma \in \mathscr{P}_n} \sigma$. S_n is an orthogonal projection and its range is called the n-fold symmetric tensor product of $\mathscr{H}$. If $\mathscr{H} = L^2(\mathbb{R}^3)$, $S_n \mathscr{H}^n$ is just the subspace of $L^2(\mathbb{R}^{3n})$ of all functions left invariant under any permutation of the variables. So we can define

Definition 2.1.2 (Symmetric (or Boson) Fock Space over $\mathscr{H}$)**.**

$$\mathscr{F}_s(\mathscr{H}) = \bigoplus_{n=0}^{\infty} S_n \mathscr{H}^n \quad .$$

Let $\epsilon : \mathscr{P}_n \to \{1, -1\}$ which is one on even permutations and minus one on odd permutations. Define $A_n = (1/n!) \sum_{\sigma \in \mathscr{P}_n} \epsilon(\sigma)\sigma$; then A_n is an orthogonal projection on $\mathscr{H}^n$. $A_n \mathscr{H}^n$ is called the n-fold antisymmetric tensor product of $\mathscr{H}$. So we define

Definition 2.1.3 (Antisymmetric (or Fermion) Fock Space over $\mathscr{H}$)**.**

$$\mathscr{F}_a(\mathscr{H}) = \bigoplus_{n=0}^{\infty} A_n \mathscr{H}^n \quad .$$

Now that we have constructed a suitable Hilbert space for a relativistic quantum system, we have to introduce the fundamental tools in describing such a system: annihilation, creation operators and quantum fields. We will do that in the Boson subspace $\mathscr{F}_s(\mathscr{H})$.

Let $f \in \mathscr{H}$ be fixed. For vectors in $\mathscr{H}^n$ of the form $\eta = \psi_1 \otimes \cdots \otimes \psi_n$[1] we define a map $b^-(f) : \mathscr{H}^n \to \mathscr{H}^{n-1}$ by[2]

$$b^-(f)\eta = (f, \psi_1)(\psi_2 \otimes \cdots \otimes \psi_n) \quad ;$$

$b^-(f)$ extends by linearity to a finite linear combination of such η, and $\|b^-(f)\eta\| \leq \|f\| \, \|\eta\|$. Thus $b^-(f)$ extends to a bounded map of $\mathscr{H}^n$ into $\mathscr{H}^{n-1}$. Since it is true for each n (except for $n = 0$ in which case we define $b^-(f) : \mathbb{C} \to 0$), $b^-(f)$ is in a natural way a bounded operator of norm $\|f\|$ from $\mathscr{F}(\mathscr{H})$ to $\mathscr{F}(\mathscr{H})$. Obviously $b^+(f) \equiv (b^-(f))^*$ takes each $\mathscr{H}^n$ into $\mathscr{H}^{n+1}$ with the action

$$b^+(f)(\psi_1 \otimes \cdots \otimes \psi_n) = f \otimes \psi_1 \otimes \cdots \otimes \psi_n$$

on product vectors. Notice that the map $f \mapsto b^+(f)$ is linear, but $f \mapsto b^-(f)$ is anti-linear.

Define $S = \bigoplus_{n=0}^{\infty} S_n$; $\mathscr{H}_s^n \equiv S_n \mathscr{H}^n$ and call it the **n-particle subspace of $\mathscr{F}_s(\mathscr{H})$**. $b^-(f)$ takes $\mathscr{F}_s(\mathscr{H})$ into itself, but $b^+(f)$ does not. A vector $\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in \mathscr{F}_s(\mathscr{H})$ for which $\psi^{(n)} = 0$ for all but finitely many n is called a **finite particle vector**. We will denote the set of finite particle vectors by F_0. The vector $\Omega_0 = \{1, 0, 0, \dots\}$ plays a special role; it is called the **vacuum**.

Let A be any self-adjoint operator on $\mathscr{H}$ with domain of essential self-adjointness D. Let $D_A = \{\psi \in F_0 | \psi^{(n)} \in \bigotimes_{k=1}^n D \text{ for each } n\}$ and define $\mathrm{d}\Gamma(A)$ on $D(A) \cap \mathscr{H}_s^n$ as

$$\mathrm{d}\Gamma(A) = A \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \mathbb{1} \otimes A \otimes \cdots \otimes \mathbb{1} + \cdots + \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes A \quad .$$

The Formalism of Second Quantization

$\mathrm{d}\Gamma(A)$ is essentially self-adjoint on D_A, and it is called the **second quantization** of A. If $A = \mathbb{1}$, then $N = \mathrm{d}\Gamma(\mathbb{1})$ is essentially self-adjoint on F_0 and for $\psi \in \mathscr{H}_s^n$, $N\psi = n\psi$. N is called the **number operator**. If U is a unitary operator on $\mathscr{H}$, we define $\Gamma(U)$ to be the unitary operator on $\mathscr{F}_s(\mathscr{H})$ which equals $\bigotimes_{k=1}^n U$ when restricted to $\mathscr{H}_s^n$ for $n > 0$, and which equals the identity on $\mathscr{H}_s^0$. If e^{itA} is a continuous unitary group on $\mathscr{H}$, then $\Gamma(e^{itA})$ is the group generated by $\mathrm{d}\Gamma(A)$, *i.e.* $\Gamma(e^{itA}) = e^{it\mathrm{d}\Gamma(A)}$.

We can now define the **annihilation operator** $a^-(f)$ on $\mathscr{F}_s(\mathscr{H})$ with domain F_0 by

$$a^-(f) = \sqrt{N+1}\, b^-(f) \quad ,$$

$a^-(f)$ is called annihilation operator because it takes each $(n+1)$-particle subspace into the n-particle subspace. For each ψ and η in F_0,

$$(\sqrt{N+1}\, b^-(f)\psi, \eta) = (\psi, S b^+(f) \sqrt{N+1}\eta) \quad .$$

This implies that

$$(a^-(f))^* \restriction F_0 = S b^+(f) \sqrt{N+1} \quad .$$

The operator $(a^-(f))^*$ is called a **creation operator**. Both $a^-(f)$ and $a^-(f)^* \restriction F_0$ are closable; we denote their closures also by $a^-(f)$ and $a^-(f)^*$.

[1] we recall that $\mathscr{H}^n$ has an orthonormal basis of such vectors.

[2] we denote with $(\cdot, \cdot)$ the $\mathscr{H}$ scalar product.

As an example consider the case $\mathscr{H} = L^2(M, \mathrm{d}\mu)$, the square integrable function on the measure space $\langle M, \mathrm{d}\mu \rangle$. We know that

$$S\bigotimes_{j=1}^{n} L^2(M, \mathrm{d}\mu) = L^2_s(M \times \cdots \times M, \mathrm{d}\mu \otimes \cdots \otimes \mathrm{d}\mu) \quad ,$$

where L^2_s is the set of functions in L^2 invariant under permutation of the coordinates. Annihilation and creation operators are then given by

$$(a^-(f)\psi)^{(n)}(m_1, \ldots, m_n) = \sqrt{n+1} \int_M \mathrm{d}\mu(m)\, \bar{f}(m)\psi^{(n+1)}(m, m_1, \ldots, m_n) \quad ,$$

$$(a^-(f)^*\psi)^{(n)}(m_1, \ldots, m_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(m_i)\psi^{(n-1)}(m_1, \ldots, \hat{m}_i, \ldots, m_n) \quad ,$$

where $\hat{m}_i$ means that m_i is omitted. If A operates on $L^2(M, \mathrm{d}\mu)$ by multiplication by the real-valued function $\omega(m)$, then

$$(\mathrm{d}\Gamma(A)\psi)^{(n)}(m_1, \ldots, m_n) = \Big(\sum_{i=1}^{n} \omega(m_i)\Big)\psi^{(n)}(m_1, \ldots, m_n) \quad .$$

It is now time to define the **Segal field operator** $\Phi_S(f)$ on F_0 as[1]

$$\Phi_S(f) = \frac{1}{\sqrt{2}}\big(a^-(f) + a^-(f)^*\big) \quad ;$$

from the definition of $a^-(f)^*$ we see that it is symmetric, in fact it is essentially self-adjoint. We will not analyze here the properties of Segal fields, see[16, 17].

2.1.2: *Q-Space*

Since we use to experience a world obeying to classical mechanics, it is usual to rephrase all physical systems in a classical-like framework. In particular we give a special role to time: we describe a system by means of "fixed-time degrees of freedom" which evolve in time. This leads to the Hamiltonian description of mechanics, both classical and quantum. We want to do that even in relativistic systems, but this breaks Lorentz invariance. The standard way to do it for a free field is to introduce the time-zero field and the canonical conjugate momentum[2]. As a matter of fact, in constructing interacting field theories by perturbing the free theory using the time-zero fields, it is quite difficult to recover Lorentz invariance. We will see that path integral formulation of field theory allows us to construct a Lorentz invariant perturbative theory in a natural and easy way.

We introduce now the so-called canonical fields starting from Segal quantization. A **conjugation** on a Hilbert space $\mathscr{H}$ is an anti-linear isometry C so that $C^2 = \mathbb{1}$.

[1] we will call the mapping $f \mapsto \Phi_S(f)$ the **Segal quantization over** $\mathscr{H}$.

[2] there is no direct connection between the canonically conjugate momentum of a field and the physical momentum operator $\widehat{P}$.

Definition 2.1.4 (canonical free field and canonical conjugate momentum)**.** Let $\mathscr{H}$ be a complex Hilbert space, $\Phi_S(\cdot)$ the associated Segal quantization. Let C be a conjugation on $\mathscr{H}$ and define $\mathscr{H}_C = \{f \in \mathscr{H} | Cf = f\}$. For each $f \in \mathscr{H}_C$ we define $\varphi(f) = \Phi_S(f)$ and $\pi(f) = \Phi_S(if)$. The map $f \mapsto \varphi(f)$ is called the **canonical free field** over $\langle \mathscr{H}, C \rangle$ and the map $f \mapsto \pi(f)$ is called the **canonical conjugate momentum**. We will drop the $\langle \mathscr{H}, C \rangle$ and just write $\mathscr{H}$, if the intended conjugation is clear. The set of elements of $\mathscr{H}$ for which the maps $f \mapsto \varphi(f)$ and $f \mapsto \pi(f)$ are defined depends on the conjugation C.

We are now ready to introduce a very useful representation of the Fock space structure we have presented: the **Q-space** and $L^2(Q, \mathrm{d}\mu)$[6, 16, 17]. Our task is to define an isomorphism S between $\mathscr{F}_s(\mathscr{H})$[1] and $L^2(Q, \mathrm{d}\mu)$ so that for each $f \in \mathscr{H}_C$, $S\varphi(f)S^{-1}$ acts on $L^2(Q, \mathrm{d}\mu)$ by multiplication by a measurable function. It will allow us to construct a complete orthonormal basis of field eigenstates. Let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$ so that each $f_n \in \mathscr{H}_C$ and let $\{g_k\}_{k=1}^{N}$ be a finite collection of the f_n. Define $\mathscr{F}_N$ as the closure of the set

$$\{P(\varphi(g_1), \ldots, \varphi(g_N))\Omega_0 | P \text{ a polynomial}\}$$

in $\mathscr{F}_s(\mathscr{H})$ and define $F_0^{(N)} = \mathscr{F}_N \cap F_0$. It follows that $\varphi(g_k)$ and $\pi(g_l)$ are essentially self-adjoint on $F_0^{(N)}$ and that

The construction of Q-space

$$e^{it\varphi(g_k)}e^{is\pi(g_l)} = e^{-ist\delta_{kl}}e^{is\pi(g_l)}e^{it\varphi(g_k)} \quad .$$

Thus we have a representation of Weyl relations[18] in which the vector Ω_0 satisfies

$$(\varphi(g_k)^2 + \pi(g_k)^2 - 1)\Omega_0 = 0$$

and is cyclic for the operators $\{\varphi(g_k)\}_{k=1}^{N}$. A Theorem by Von Neumann[19, 20] therefore guarantees us there is a unitary map $S_1^{(N)} : \mathscr{F}_N \to L^2(\mathbb{R}^N, \mathrm{d}\mu_1)$ (where $\mathrm{d}\mu_1 \equiv \mathrm{d}^N x$, the Lebesgue measure of $\mathbb{R}^N$), so that

$$S_1^{(N)}\varphi(g_k)(S_1^{(N)})^{-1} = x_k \quad ,$$
$$S_1^{(N)}\pi(g_k)(S_1^{(N)})^{-1} = \frac{1}{i}\partial_k \quad ,$$

and the vacuum corresponds to the vector

$$S_1^{(N)}\Omega_0 = \pi^{-\frac{N}{4}} \exp\left\{-\sum_{k=1}^{N} \frac{x_k^2}{2}\right\} \quad .$$

Often it is convenient that the transform of the vacuum is the function identically one, so one introduces the Hilbert space

$$L^2(\mathbb{R}^N, \pi^{-\frac{N}{2}} \exp\{-\textstyle\sum x_k^2/2\}\mathrm{d}^N x)$$

[1] it is possible to do the same construction in the Fermionic case $\mathscr{F}_a(\mathscr{H})$, but since we will not make use fermionic fields in this work we do not develop such a construction.

instead of $L^2(\mathbb{R}^N)$ so let $\mathrm{d}\mu_{k,2} = \pi^{-1/2} e^{-x_k^2} \mathrm{d}x_k$ and define

$$(Tf)(x) = \pi^{\frac{N}{4}} \exp\left\{ \sum_{k=1}^{N} \frac{x_k^2}{2} \right\} f(x) \quad .$$

Then T is a unitary map of $L^2(\mathbb{R}^N, \mathrm{d}\mu_1)$ onto $L^2(\mathbb{R}^N, \mathrm{d}\mu_2)$ with $\mathrm{d}\mu_2 = \prod_{k=1}^{N} \mathrm{d}\mu_{2,k}$ and if we define $S_2^{(N)} = T S_1^{(N)}$, we have $S_2^{(N)} : \mathscr{F}_N \to L^2(\mathbb{R}^N, \mathrm{d}\mu_2)$, and

$$\begin{aligned} S_2^{(N)} \varphi(g_k) (S_2^{(N)})^{-1} &= x_k \quad , \\ S_2^{(N)} \pi(g_k) (S_2^{(N)})^{-1} &= -\frac{x_k}{i} + \frac{1}{i} \partial_k \quad , \\ S_2^{(N)} \Omega_0 &= 1 \quad . \end{aligned}$$

Since each $\mu_{2,k}$ has mass one, the following property holds for both maps $S_1^{(N)}$ and $S_2^{(N)}$, so we will omit the index $_1$ or $_2$[1]:

$$\begin{aligned} (\Omega_0, P_1(\varphi(g_1)) \cdots P_N(\varphi(g_N))\Omega_0) = \int_{\mathbb{R}^N} \mathrm{d}\mu \, P_1(x_1) \cdots P_N(x_N) &= \prod_{k=1}^{N} \int_{\mathbb{R}} \mathrm{d}\mu_k \, p_k(x_k) \\ &= \prod_{k=1}^{N} (\Omega_0, P_k(\varphi(g_k))\Omega_0) \quad , \end{aligned}$$

where $P_1, \ldots, P_N$ are polynomials.

We can now construct $\langle Q, \mathrm{d}\mu \rangle$. We define $Q = \times_{k=1}^{\infty} \mathbb{R}$. Taking the σ-algebra generated by countable products of measurable sets in $\mathbb{R}$ we set $\mu = \bigotimes_{k=1}^{\infty} \mu_k$. The points of Q will be denoted by $q = \langle q_1, q_2, \ldots \rangle$. Then $\langle Q, \mu \rangle$ is a measure space and the set of functions of the form $P(q_1, \ldots, q_n)$, where P is a polynomial and n is arbitrary, is dense in $L^2(Q, \mathrm{d}\mu)$. Let P be a polynomial in N variables:

$$P(x_{k_1}, \ldots, x_{k_n}) = \sum_{l_1, \ldots, l_N} c_{l_1 \ldots l_M} x_{k_1}^{l_1} \cdots x_{k_N}^{l_N} \quad ,$$

and define

$$S : P(\varphi(f_{k_1}), \ldots, \varphi(f_{k_N}))\Omega_0 \to P(q_{k_1}, \ldots, q_{k_N}) \quad .$$

Then by the property above

$$\begin{aligned} \left\| P(\varphi(f_{k_1}), \ldots, \varphi(f_{k_N}))\Omega_0 \right\|^2 &= \sum_{l,m} c_l \bar{c}_m (\Omega_0, \varphi(f_{k_1})^{l_1+m_1} \cdots \varphi(f_{k_N})^{l_N+m_N} \Omega_0) \\ &= \sum_{l,m} c_l \bar{c}_m \int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}\mu_{k_i} \, q_{k_1}^{l_1+m_1} \cdots q_{k_N}^{l_N+m_N} = \int_Q \mathrm{d}\mu \, \left| P(q_{k_1}, \ldots, q_{k_N}) \right|^2 \quad . \end{aligned}$$

Since Ω_0 is cyclic for the polynomials in the fields, S extends to a unitary map of $\mathscr{F}_s(\mathscr{H})$ onto $L^2(Q, \mathrm{d}\mu)$. Obviously $S\varphi(f_k)S^{-1} = q_k$, and $S_2\Omega_0 = 1$, while

$$S_1\Omega_0 \propto \exp\left\{ -\sum\nolimits_k \frac{q_k^2}{2} \right\} \quad .$$

[1] we will do the same in the following: when we don't explicit the index it means that the construction is valid in both cases.

So we have proved that $\mathscr{F}_s(\mathscr{H})$ is isomorphic to a space of square integrable functions where the canonical field operator reduces to multiplication by a coordinate and its conjugate momentum to differentiation with respect to the same coordinate: so they behave as position and momentum operators on the space of square integrable functions isomorphic to $\mathscr{H}$; but since we can find an orthonormal basis of position eigenstates in $\mathscr{H}$, and also a basis of momentum eigenstates, that means we can find an orthonormal basis of canonical time zero field eigenstates, or conjugate momentum ones, in $\mathscr{F}_s(\mathscr{H})$. Explicitly, with the usual Dirac notation, we have a continuous basis $\{|q\rangle\} \in \mathscr{F}_s(\mathscr{H})$ so that $\varphi(f_k)|q\rangle = q_k|q\rangle$, $\mathbb{1} = \int \mathrm{d}\mu_1(q)\,|q\rangle\langle q|$; and a basis $\{|p\rangle\} \in \mathscr{F}_s(\mathscr{H})$ so that $\pi(f_k)|p\rangle = p_k|p\rangle$, $\mathbb{1} = \int \mathrm{d}\mu_1(p)\,|p\rangle\langle p|$; furthermore we know that

$$\langle q|p\rangle = \prod_k \frac{1}{(2\pi)^{1/2}} e^{iq_k p_k} \quad ,$$

and if we call the vacuum vector $|0\rangle$,

$$\langle q|0\rangle \propto \exp\left\{-\sum_k \frac{q_k^2}{2}\right\} \quad .$$

2.1.3: *Derivation of Path integral*

In order to find a path integral formulation of perturbative quantum field theory we have to apply the useful mathematical results of previous sections on physical situations. Usually in quantum field theory one uses the annihilation operator[1] $a(p)$ on $\mathscr{F}_s(L^2(\mathbb{R}^3))$, $p \in \mathbb{R}^3$, the free scalar field and time zero fields thus becoming[4]:

$$\Phi_m(x,t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\mu(p))^{1/2}} \left(e^{i(\mu(p)t - p\cdot x)} a^\dagger(p) + e^{-i(\mu(p)t - p\cdot x)} a(p)\right) \quad ,$$

$$\varphi_m(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\mu(p))^{1/2}} \left(e^{-ip\cdot x} a^\dagger(p) + e^{ip\cdot x} a(p)\right) \quad ,$$

$$\pi_m(x) = \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \sqrt{\frac{\mu(p)}{2}}\, \mathrm{d}^3 p \left(e^{-ip\cdot x} a^\dagger(p) + e^{ip\cdot x} a(p)\right) \quad ,$$

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with $\mu(p) = \sqrt{p^2 + m^2}$. We can repeat the construction of the Q-space for these canonical fields, but the Q-space has to be a functional space, in particular the action of S_1(the unitary transformation we will use) would become

$$S_1 \varphi_m(x) S_1^{-1} = q(x) \quad ,$$
$$S_1 \pi_m(x) S_1^{-1} = \frac{1}{i} \frac{\delta}{\delta q(x)} \quad ,$$

and since we can write

$$S_1 a(p) S_1^{-1} = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 x\, e^{-ip\cdot x} \left(\sqrt{\frac{\mu(p)}{2}} q(x) + \sqrt{\frac{1}{2\mu(p)}} \frac{\delta}{\delta q(x)}\right) \quad ,$$

[1] for the sake of simplicity we consider the case of an Hermitian scalar field of mass m, the general to higher spin fields is straightforward.

and $S_1 a(p) S_1^{-1} S_1 \Omega_0 = 0$, we obtain that

$$S_1\Omega_0 = \mathcal{N}\exp\Big\{-\frac{1}{2}\int \mathrm{d}^3x\mathrm{d}^3y\, \mathcal{E}(x,y)q(x)q(y)\Big\} \quad ,$$

where $\mathcal{N}$ is a constant and $\mathcal{E}(x,y) = (2\pi)^{-3}\int \mathrm{d}^3p\, e^{ip\cdot(x-y)}\mu(p)$. We can again construct a basis of field or momentum eigenstates, but now they will satisfy the following relations[1]:

$$\begin{aligned}
\varphi_m(x)|q\rangle &= q(x)|q\rangle \quad , \\
\pi_m(x)|p\rangle &= p(x)|p\rangle \quad , \\
\mathbb{1} = \int \mathscr{D}[q(x)]|q\rangle\langle q| &= \int \mathscr{D}[p(x)]|p\rangle\langle p| \quad , \\
\langle q|p\rangle &= \mathscr{M}\exp\Big\{i\int \mathrm{d}^3x\, q(x)p(x)\Big\} \quad , \\
\langle q|0\rangle &= \mathcal{N}\exp\Big\{-\frac{1}{2}\int \mathrm{d}^3x\mathrm{d}^3y\, \mathcal{E}(x,y)q(x)q(y)\Big\} \quad ,
\end{aligned} \tag{2.1.1}$$

where $\mathscr{M}$ and $\mathcal{N}$ are constants.

We know that to obtain elements of the S-matrix we have to know the vacuum expectation values:

$$\langle\, \varphi(x_1,t_1)\varphi(x_2,t_2)\cdots\rangle_0 \equiv \frac{\langle 0,\mathrm{out}|\mathcal{T}\Big\{\varphi(x_1,t_1)\varphi(x_2,t_2)\cdots\Big\}|0,\mathrm{in}\rangle}{\langle 0,\mathrm{out}|0,\mathrm{in}\rangle} \quad ;$$

but we can write both numerator and denominator in a path integral form using relations (2.1.1) and a derivation identical to the quantum mechanical one developed in Sections 1.2 and 1.3. So we obtain a formula[2]:

$$\begin{aligned}
\langle 0,\mathrm{out}|\mathcal{T}\Big\{\varphi(x_1,t_1)\varphi(x_2,t_2)\cdots\Big\}|0,\mathrm{in}\rangle = \int \mathscr{D}[q(x,t)]\mathscr{D}[p(x,t)]q(x_1,t_1)q(x_2,t_2)\cdots \\
\cdots\cdots\cdots\exp\Big\{i\Big[\int_{-\infty}^{\infty}\mathrm{d}t\int \mathrm{d}^3x\,\big(\dot{q}(x,t)p(x,t) - H(q(x,t),p(x,t))\big) \\
+\frac{i}{2}\int \mathrm{d}^3x\mathrm{d}^3y\,\mathcal{E}(x,y)\big(q(x,+\infty)q(y,+\infty) + q(x,-\infty)q(y,-\infty)\big)\Big]\Big\},
\end{aligned}$$

where $H(q(x,t),p(x,t))$ is the fields Hamiltonian density, and the last term is due to the projections $\langle 0,\mathrm{out}|q_{+\infty}\rangle$ and $\langle q_{-\infty}|0,\mathrm{in}\rangle$. For any reasonably smooth function $g(x)$ holds the following equality:

$$g(+\infty) + g(-\infty) = \lim_{\epsilon\to 0^+}\ \epsilon\int_{-\infty}^{\infty}\mathrm{d}t\, g(t)e^{-\epsilon|t|} \quad ;$$

so we have

$$\frac{i}{2}\int \mathrm{d}^3x\mathrm{d}^3y\,\mathcal{E}(x,y)\big(q(x,+\infty)q(y,+\infty) + q(x,-\infty)q(y,-\infty)\big) = i\epsilon\int_{-\infty}^{\infty}\mathrm{d}t\,\mathcal{G}(t) \quad ,$$

[1] $\mathscr{D}[x(t)]$ is the measure associated with the Q-space in such a case; since the Q-space is a functional space, the measure would be a functional measure.

[2] the constants $\mathscr{M}$ and $\mathcal{N}$ are adsorbed in the definition of a proper functional measure for time-dependent fields.

with $\mathcal{G}(t) = 1/2 \int \mathrm{d}^3x \mathrm{d}^3y\, \mathcal{E}(x,y) q(x,t) q(y,t)$ a quadratic form in the fields. So finally we obtain the phase space path integral for a quantum field theory:

$$\langle 0, \mathrm{out}|\mathcal{T}\left\{\varphi(x_1,t_1)\varphi(x_2,t_2)\cdots\right\}|0,\mathrm{in}\rangle = \int \mathcal{D}[q(x,t)]\mathcal{D}[p(x,t)] q(x_1,t_1) q(x_2,t_2)\cdots \\ \cdots \exp\left\{ i\int_{-\infty}^{\infty} \mathrm{d}t \left[\int \mathrm{d}^3x \left(\dot{q}(x,t)p(x,t) - H\big(q(x,t),p(x,t)\big)\right) + i\epsilon \mathcal{G}(t)\right]\right\}. \tag{2.1.2}$$

The whole effect of the last term in Eq. (2.1.2) is to provide the Feynman prescription in the field propagator, solving the ambiguities due to poles on integration path. Following the procedure of Section 1.2.3 we can pass from phase space path integral to configuration space path integral, but if the factor $\mathrm{Det}\,\mathscr{A}[q]$ is field dependent it gives a correction modifying the effective Lagrangian density, and this modification has to be calculated case by case. However in most theories $\mathrm{Det}\,\mathscr{A}$ is field-independent, so it can be adsorbed in the definition of functional measure without problems. Moreover all this construction, from the definition of Q-space to Eq. (2.1.2), is generalized to vectorial fields without any further technical difficulty, and also to fermionic fields, introducing the Grassman algebra of commuting variables and Berezinian calculus[1]; so we can write the configuration space path integral for an arbitrary field:

$$\langle 0, \mathrm{out}|\mathcal{T}\left\{\Psi_{l_1}(x_1,t_1)\Psi_{l_2}(x_2,t_2)\cdots\right\}|0,\mathrm{in}\rangle = \int \mathcal{D}[\psi_l(x,t)]\psi_{l_1}(x_1,t_1)\psi_{l_2}(x_2,t_2)\cdots e^{iS[\psi]} \tag{2.1.3}$$

QFT Path Integral

where $S[\psi]$ is the field action

$$S[\psi] = \int_{-\infty}^{\infty} \mathrm{d}t \Big[\int \mathrm{d}^3x\, \mathscr{L}_{eff}(\psi_l(x,t), \dot{\psi}_l(x,t)) + i\epsilon \mathcal{G}(t)\Big] \quad ,$$

and the covariant Lagrangian density $\mathscr{L}_{eff}$ takes into account possible corrections due to $\mathrm{Det}\,\mathscr{A}[\psi]$. It is possible now to derive free propagators, Wick theorem and Feynman rules in a way analogous to the quantum mechanical one we developed in the previous chapter, however further calculations are beyond the scope of this work. Nevertheless we emphasize that since propagators and vertex contributions in the Feynman rules are obtained directly by inspection of the Lagrangian, explicitly covariant, we avoid all the problems due to non-covariant nature of Hamiltonian formulation, where propagator and interacting Hamiltonian both contain non-covariant terms[2]. Finally we can say that path integrals provide a beautiful method to perform explicit quantum field theory calculations that go directly from the covariant canonical Lagrangian to the Feynman rules in their final, Lorentz-covariant form[13].

[1] since we will not use fermionic path integrals in quantum field theory we will not analyze these mathematical structures here.

[2] obviously even in the usual operator formulation these terms cancel out in order to restore Lorentz-covariance of the theory.

2.2: *Worldline Methods*

Although we did not perform explicit calculations in the previous section, we mentioned that, as usual in path integrals, we find free propagators introducing external sources in the generating functional for the free action, and in field theory these source terms are equivalent to a coupling of the free field to external classical currents. We will see in a moment that it is very useful to define a generating functional even for the complete action, for it will lead to the introduction of the so-called "quantum effective action". This effective action $\Gamma[\psi]$ will allow us to take into account all multi-loop effects by summing 'tree' graphs, *i.e.* without loops, whose vertices and propagators are taken from $\Gamma[\psi]$. Furthermore we will see that we can calculate the one-loop part of such effective actions by means of a path integral in a suitable "fictitious" quantum mechanics.

2.2.1: *The Quantum Effective Action*

Consider a field theory with action $S[\psi]$, with $\psi^l(x)$[1] a generic field (however we shall not bother to keep track of the minus signs that would appear in that case). We introduce a set of classical currents $J_l(x)$ coupled to the fields $\psi^l(x)$. So we define the generating functional

$$Z[J] \equiv \langle 0, \text{out}|0, \text{in}\rangle_J = \int \mathscr{D}[\psi^l(y)]\, e^{iS[\psi]+i\int \mathrm{d}^4x\, \psi^r(x)J_r(x)} \quad . \tag{2.2.1}$$

Now, $Z[J]$ is given by the sum of all vacuum-vacuum amplitudes in the presence of the current J, including disconnected as well as connected diagrams. In order to take into account only connected diagrams we introduce

$$W[J] \equiv \frac{1}{i}\ln Z[J] \quad .$$

So in terms of the sum of connected graphs we have

$$\psi^l_j(x) \equiv \frac{\langle 0, \text{out}|\Psi^l(x)|0, \text{in}\rangle_J}{\langle 0, \text{out}|0, \text{in}\rangle_J} = \frac{1}{iZ[J]}\frac{\delta}{\delta J_l(x)}Z[J] = \frac{\delta}{\delta J_l(x)}W[J] \quad .$$

Such a formula can be inverted, in the sense that we can invert the map $J_l(x) \mapsto \psi^l_J(x)$ defining a map $\psi^l(x) \mapsto J_{l,\psi}(x) \mid \psi^l(x) = (\delta/\delta J_{l,\psi}(x))W[J]$. Now we are able to define the **quantum effective action** $\Gamma[\psi]$ as the Legendre transformation

$$\Gamma[\psi] \equiv W[J_\psi] - \int \mathrm{d}^4x\, \psi^l(x)J_{l,\psi}(x) \quad .$$

$\Gamma[\psi]$ is the sum of all connected *one-particle-irreducible*[2] graphs.

From its own definition we can explain why it is called the 'effective action': if we calculate its functional derivative

$$\frac{\delta\Gamma[\psi]}{\delta\psi^r(y)} = -J_{r,\psi}(y) \quad ;$$

[1] we use a shorthand notation where x is the four-dimensional space-time vector.

[2] A one-particle-irreducible ($1PI$) graph is one that cannot be disconnected by cutting through any one internal line.

we see that the possible values for the external fields $\psi^r(y)$ in the absence of currents J are given by the *stationary points* of Γ:

$$\frac{\delta\Gamma[\psi]}{\delta\psi^r(y)} = 0 \quad .$$

Hence this may be regarded as the equation of motion for the external field ψ, taking quantum corrections into account.

However $\Gamma[\psi]$ does not provide only the quantum-corrected field equations; it can be shown that $iW[J]$ can be calculated as a sum of connected tree graphs, with vertices calculated as if the action were $\Gamma[\psi]$ instead of $S[\psi]$. We don't give here the rigorous proof, however it is possible to show that[21]:

$$iW[J] = \int\limits_{\substack{\text{connected,}\\ \text{tree graphs}}} \mathscr{D}[\psi^l(x)] e^{i\Gamma[\psi] + i\int \mathrm{d}^4y\, \psi^r(y) J_r(y)} \quad ,$$

where the boundary means that we have to keep only connected, tree graphs. In order for this last expression to be correct, $i\Gamma[\psi]$ must be the sum of all one-particle-irreducible connected graphs with arbitrary number of external lines, each line corresponding to a factor ψ rather than a propagator or wave function. For this reason $i\Gamma[\psi_0]$ for some fixed field $\psi_0^l(x)$ may be expressed as the sum of one-particle-irreducible graphs for the vacuum-vacuum amplitude, calculated with the action $S[\psi + \psi_0]$:

Quantum Effective Action and its applications

$$i\Gamma[\psi_0] = \int\limits_{\substack{1PI\\ \text{connected}}} \mathscr{D}[\psi^l(x)] e^{iS[\psi+\psi_0]} \quad .$$

Differentiating Γ with respect to ψ N times we obtain the N-point correlation function $1PI$. If we consider the complete 2-point correlation function (or complete propagator), we have

$$\Delta^{lr}(x,y) \equiv \frac{\delta^2 W[J]}{\delta J_l(x)\delta J_r(y)} = \frac{\delta\psi_J^l(x)}{\delta J_r(y)} \quad ,$$

while its $1PI$ part $\Pi_{lr}(x,y)$ is given by

$$\Pi_{lr}(x,y)\frac{\delta^2\Gamma[\psi]}{\delta\psi^l(x)\delta\psi^r(y)} = -\frac{\delta J_{l,\psi}(x)}{\delta\psi^r(y)} \quad ;$$

it follows immediately that the matrices Δ and Π are related by $\Delta = -\Pi^{-1}$, and this is the well-known relation between propagators and self-energy parts.

It is useful to split the effective action in its contributions due to $1PI$ connected graphs with zero, one, ... loops. In the case of a self-interacting real scalar field $\phi(x)$ with action

$$S[\phi] = -\int \mathrm{d}^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{24}g\phi^4\right] \quad ,$$

and $\phi_0(x) = \phi_0$ a constant field, we obtain

$$i\Gamma^{0\text{ loop}}[\phi_0] = -i\mathcal{V}_4\left(\frac{1}{2}m^2\phi_0^2 + \frac{g}{24}\phi_0^4\right) \quad ,$$
$$i\Gamma^{1\text{ loop}}[\phi_0] = -\frac{1}{2}\,\mathrm{Tr}\ln\left(\frac{iK}{\pi}\right) \quad ,$$

where $\mathcal{V}_4$ is the (divergent) volume of space-time, and

$$K = -\Box + \mu^2(\phi_0) - i\epsilon \quad ,$$

with $\Box = \nabla^2 - \partial_t^2$ and $\mu^2(\phi_0) = m^2 + (1/2g)\phi_0^2$. Note that the one-loop effective action is written as the trace of the logarithm of an operator: the presence of such a logarithm will allow us to write it as a quantum mechanics path integral.

Consider another example: if we have an action $S[\psi, A]$, where A is an external background field, that is a (purely) quadratic function of ψ the vertices of the theory would contain only two ψs and a certain number of As (usually one, or maybe two); but since A could appear only as an external leg in the graphs since it is a background field, the only connected diagrams are one-loop ψ diagrams with an arbitrary number of A insertions. In the case of one-loop diagrams the connected part is also $1PI$, and the insertion of A gives analytically a multiplicative factor A; so if we write in Euclidean time

$$Z[A] = \langle 0, \text{out}|0, \text{in}\rangle_A = \int \mathcal{D}[\psi] e^{-S_E[\psi,A]} = e^{-W[A]} \quad ,$$

we recognize that $\Gamma^{1\text{ loop}}[A] = W[A]$, the connected part of $\langle 0, \text{out}|0, \text{in}\rangle_A$. But since $S_E[\psi, A]$ is a purely quadratic form we can write it in a condensed notation as

$$S_E[\psi, A] = \int \mathrm{d}^4x\, \psi^r K^A_{rs} \psi^s \quad ,$$

but in that case the path integral in $Z[A]$ is a usual Gaussian integral, so we solve it obtaining

$$Z[A] = \mathcal{C}\,(\mathrm{Det}\, K^A)^\rho \quad ,$$

where $\rho = -1/2, -1$ for a system of real or complex bosonic fields, and $\rho = 1$ for a system of fermionic fields. So since $\ln \mathrm{Det}\,\mathscr{A} = \mathrm{Tr}\ln\mathscr{A}$ we can write[1]:

$$\Gamma^{1\text{ loop}}[A] = -\rho \ln \mathrm{Det}\, K^A \quad ; \tag{2.2.2}$$

again we see that the one-loop effective action is written as the trace of the logarithm of an operator.

2.2.2: *Fictitious Quantum Mechanics*

We are now able to introduce the beautiful worldline method to calculate one-loop effective actions. The starting-point is the idea, due to Schwinger[22], that we can use the formula

$$\ln\left(\frac{A}{B}\right) = -\int_0^\infty \frac{\mathrm{d}T}{T}\left(e^{-AT} - e^{-BT}\right)$$

[1] dropping the non-influent, A-independent additive constant $\ln \mathcal{C}$.

to rewrite the one-loop effective action of Eq. (2.2.2)[1] as:

$$\Gamma^{1\,\mathrm{loop}}[A] = -\rho \ln \mathrm{Det}\, K^A = \rho \int_0^\infty \frac{\mathrm{d}T}{T} \mathrm{Tr}\, e^{-TK^A} \quad ,$$

where we have dropped an additive, A-independent constant[2]. Now if we redefine $K^A \equiv \mathcal{H}[A]$, $T \equiv \beta$, we obtain, remembering Eq. (1.3.19)

$$\Gamma^{1\,\mathrm{loop}}[A] = \rho \int_0^\infty \frac{\mathrm{d}\beta}{\beta} \mathcal{Z}[\beta] = \rho \int_0^\infty \frac{\mathrm{d}\beta}{\beta} \int_{PBC} \mathscr{D}[x(t)] e^{-S_{E,A}[x,\beta]} \quad , \tag{2.2.3}$$

One-loop Effective Action as a path integral

where $S_{E,A}[x,\beta]$ is the Euclidean action of a fictitious quantum mechanical system with Hamiltonian function $\mathcal{H}_A$. So if we know the classical action corresponding to a quantum Hamiltonian function $\mathcal{H}_A$ we are able to calculate the one-loop effective action as a quantum mechanical path integral.

In the previous section we found that the one-loop effective action for the constant part of a self-interacting scalar field with quartic interaction was

$$i\Gamma^{1\,\mathrm{loop}}[\phi_0] = -\frac{1}{2} \mathrm{Tr} \ln\Big(\frac{iK}{\pi}\Big) \quad ,$$

in Euclidean time it becomes

$$\Gamma^{1\,\mathrm{loop}}[\phi_0] = -\frac{1}{2} \mathrm{Tr} \ln\Big(\frac{\mathcal{H}_{E,\phi_0}}{\pi}\Big) \quad ,$$

where $\mathcal{H}_{E,\phi_0} = -\nabla_4^2 + \mu^2(\phi_0)$, with ∇_4^2 the 4-dimensional Laplacian operator. We know that this is the Hamiltonian function of a 4-dimensional particle of mass $1/2$ moving in a constant potential $\mu^2(\phi_0)$. So in that case Eq. (2.2.3) becomes

$$\Gamma^{1\,\mathrm{loop}}[\phi_0] = -\frac{1}{2} \int_0^\infty \frac{\mathrm{d}\beta}{\beta} \int_{PBC} \mathscr{D}[x(t)] \exp\Big\{ -\int_0^\beta \mathrm{d}t \, \big(\frac{1}{4}\dot{x}(t)^2 + \mu^2(\phi_0)\big) \Big\} \quad ;$$

thus we have seen that the one-loop effective action is calculated by means of a fictitious 4-dimensional spinless particle moving in a constant potential.

We have certainly emphasized that the quantum mechanics we use to describe the effective action is a fictitious one, but a careful analysis of worldline models for relativistic particles will reveal a quite interesting feature: this fictitious quantum mechanics is not so fictitious, for it corresponds to the first quantization of the relativistic particle which makes the loop in the effective action[3]. However we will not analyze worldline models and their implications here; we will do it in the next chapter in the more general case of curved spacetime. For a quite complete review of such methods see[23].

[1] or in general every time it is written as the trace of a logarithm.

[2] although this constant does not affect the calculation of correlation functions or of the equations of motion for A and is thus canceled, it is *infinite*!

[3] in the case, as analyzed in the previous section, of a field with a quadratic action coupled to an external background field; in that case the connected diagrams reduces to one-loop diagrams with an arbitrary number of background field insertions.

Appendices

Appendix A

Trotter formula and other systematic approximants

We introduce here the Trotter formula and its range of applicability, and mention some other methods to approximate the exponential of a sum of operators by means of product of exponentials.

A.1: Trotter product formula

The trotter product formula is the most used to express $\exp\{it(A+B)\}$ as a product of exponentials since it holds for a wide class of operators A and B; in fact we can formulate it under the general assumption

Theorem A.1.1 (Trotter product formula). *If A and B are self-adjoint operators and $A+B$ is essentially self-adjoint on $D(A)\cap D(B)$, then*

$$\mathrm{s}-\lim_{n\to\infty}(e^{itA/n}e^{itB/n})^n = e^{i(A+B)t} \quad .$$

Moreover, if A and B are bounded from below, then

$$\mathrm{s}-\lim_{n\to\infty}(e^{-tA/n}e^{-tB/n})^n = e^{-(A+B)t} \quad .$$

We don't proof this general theorem here, see[24].

This formula applies to path integral only if we could write the Hamiltonian as a sum of two or more parts; however in Section 1.2.1 we showed that we can find a similar formula, that holds at first order in t/n, for the normal ordering of $e^{i\epsilon\mathcal{H}}$ (that can be approximated with the exponential of the normal ordered Hamiltonian), and in Appendix D we will do a similar approximation for the Weyl ordering of $e^{-\epsilon\mathcal{H}}$.

A.2: *Zassenhaus formula*

In the formulation of path integral Trotter formula or a first order approximation in t/n is sufficient, but if we wanted to evaluate the discretized action at orders higher than the first in ϵ, as it is sometimes done in statistical mechanics, we have to use the Zassenhaus formula[8, 9]:

$$e^{\lambda(A+B)} = e^{\lambda A} e^{\lambda B} e^{\lambda^2 C_2} \cdots e^{\lambda^m C_m} \cdots \quad ,$$

where

$$C_n = \frac{1}{n!} \left[\frac{\partial^n}{\partial \lambda^n} \left(e^{-\lambda^{n-1} C_{n-1}} \cdots e^{-\lambda B} e^{-\lambda A} e^{\lambda(A+B)} \right) \right] \Bigg|_{\lambda=0} \quad ,$$

so as an example

$$C_2 = \frac{1}{2}[B, A] \, , \quad C_3 = \frac{1}{6}[C_2, A + 2B] \, .$$

However this formula is proved to approximate the operator $e^{\lambda(A+B)}$ at the desired order in a rigorous way only if A and B are bounded, so it is not useful with an Hamiltonian with unbounded kinetic part.

Appendix B

the Dyson expansion

In this appendix we will give a proof of Theorem 1.2.1, and discuss briefly its range of applicability. We will state here a useful theorem:

Theorem B.0.1 (Principle of uniform boundedness). *Let X be a Banach space, $\mathscr{F}$ a family of bounded linear transformations from X to some normed linear space Y. If, for each $x \in X$, $\{\|Tx\|_y \mid T \in \mathscr{F}\}$ is bounded; then $\{\|T\| \mid T \in \mathscr{F}\}$ is bounded.*

We also recall that we can construct a Banach space $\mathscr{L}(X,Y)$ of operators from a Banach space to another, if we define a norm:

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X} \quad ,$$

the induced topology on $\mathscr{L}(X,Y)$ is called **uniform operator topology**.

B.1: Proof of the Theorem

For the sake of commodity we will repeat here Theorem 1.2.1:

Theorem B.1.1 (the Dyson expansion). *Let $t \to \mathcal{H}(t)$ be a strongly continuous map of $\mathbb{R}$ into the* bounded *self-adjoint operators on a Hilbert space $\mathscr{H}$. Then there is a unitary propagator $\mathcal{U}(t,s)$ on $\mathscr{H}$ so that, for all $\psi \in \mathscr{H}$,*

$$\varphi_s(t) = \mathcal{U}(t,s)\psi \tag{B.1.1}$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_s(t) = -i\mathcal{H}(t)\varphi_s(t)\,, \quad \varphi_s(s) = \psi \quad . \tag{B.1.2}$$

Proof. Let's start defining Dyson expansion:

$$\mathcal{U}(t,s)\varphi = \varphi + \sum_{n=1}^{\infty}(-i)^n \int_s^t \int_s^{t_1} \cdots \int_s^{t_{n-1}} \mathrm{d}t_1 \cdots \mathrm{d}t_n \, \mathcal{H}(t_1)\cdots\mathcal{H}(t_n)\varphi \quad ; \tag{B.1.3}$$

$\mathcal{H}(\tau)$ is uniformly bounded on $[s,t]$ by Theorem B.0.1, so the nth term on the right is bounded by

$$\frac{|t-s|^n}{n!}\left(\sup_{\tau\in[s,t]}\|\mathcal{H}(\tau)\|\right)^n \|\varphi\|$$

so the series on the right converges in the uniform operator topology to $\mathcal{U}(t,s)$. Thus $\mathcal{U}(t,s)$ is jointly strongly continuous in s and t since this is true of each term on the right. It is trivial to check that $\mathcal{U}(t,t) = \mathbb{1}$ and $\mathcal{U}^\dagger(t,s) = \mathcal{U}(s,t)$. The formula $\mathcal{U}(r,s)\mathcal{U}(s,t) = \mathcal{U}(r,t)$ is proven by multiplying out the series. Therefore

$$\mathcal{U}(s,t)\mathcal{U}^\dagger(s,t) = \mathbb{1} = \mathcal{U}^\dagger(s,t)\mathcal{U}(s,t) \quad ,$$

so $\mathcal{U}(t,s)$ is unitary. The first statement of Eq. (B.1.2) is proven differentiating the Dyson series for $\mathcal{U}(t,s)$ term by term and noting that the resulting series converges uniformly. □

Finally we state a more general theorem that extends (B.1.3) to the case of contraction semi-groups on a Banach space[6].

Definition B.1.1 (Contraction semi-group)**.** A family $\{T(t)|0 \le t < \infty\}$ of bounded operators on a Banach space X is called a **contraction semi-group** if

1. $T(0) = \mathbb{1}$,
2. $T(s)T(t) = T(s+t)$ for all $s,t \in \mathbb{R}^+$,
3. for each $\varphi \in X$, $t \mapsto T(t)\varphi$ is continuous;

and moreover $\|T(t)\| \le 1$ for all $t \in [0,\infty)$.

We can write $T(r) = e^{-rA}$, where A is said to be the generator of the contraction semi-group.

Now consider a generator $A(t)$ of a contraction semi-group, and define

$$U_k(t,s) = \exp\left\{-(t-s)A\left(\frac{i-1}{k}\right)\right\} \quad \text{with } i,k \text{ integers, } 1 \le i \le k,$$
$$\text{and } t,s \in \mathbb{R}^+ \text{ with } \frac{i-1}{k} \le s \le t \le \frac{i}{k} \quad .$$

Finally define $C(t,s) \equiv A(t)A(s)^{-1} - \mathbb{1}$, and we have the following theorem:

Theorem B.1.2. *Let X be a Banach space and let I be an open interval in $\mathbb{R}$. For each $t \in I$, let $A(t)$ be the generator of a contraction semi-group on X so that 0 is in the resolvent set of $A(t)$ and*

(i) the $A(t)$ have a common domain D (from which it follows that $A(t)A(s)^{-1}$ is bounded);

(ii) for each $\varphi \in X$, $(t-s)^{-1}C(t,s)\varphi$ is uniformly strongly continuous and uniformly bounded in s and t for $t \neq s$ lying in any fixed subinterval of I;

(iii) for each $\varphi \in X$, $C(t)\varphi \equiv \lim_{s \nearrow t}(t-s)^{-1}C(t,s)\varphi$ exists uniformly for t in each compact subinterval and $C(t)$ is bounded and strongly continuous in t.

Then for all $s \leq t$ in any compact subinterval of I and any $\varphi \in X$,

$$U(t,s)\varphi = \lim_{k\to\infty} U_k(t,s)\varphi$$

exists uniformly in s and t. Further, if $\psi \in D$, then $\varphi_s(t) \equiv U(t,s)\psi$ is in D for all t and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_s(t) = -A(t)\varphi_s(t), \quad \varphi_s(s) = \psi \quad ;$$

and $\|\varphi_s(t)\| \leq \|\psi\|$ for all $t \geq s$.

Appendix C

Gaussian integrals

Throughout the work we make large use of Gaussian integrals; we discuss them here. The most general Gaussian integral can be written as

$$\mathscr{I} = \int_{-\infty}^{\infty} \prod_r \mathrm{d}\zeta_r \, e^{-Q(\zeta)} \quad ,$$

where $Q(\zeta)$ is a general quadratic function of ζ:

$$Q(\zeta) = \frac{1}{2} K^{rs} \zeta_r \zeta_s + L^r \zeta_r + M \quad ,$$

with the usual summation convention of repeated indices. The matrix $\mathbf{K}$ has to be symmetric and non-singular. We consider so the case where K^{rs}, L^r and M are all real, with K^{rs} also positive; the result in the general case can then be obtained by analytic continuation.

A symmetric real matrix can be diagonalized by an orthogonal matrix, therefore there is a matrix S, $S^T = S^{-1}$ such that

$$\left(S^T K S\right)^{rs} = \delta^{rs} k_r \quad .$$

The eigenvalues k_r are positive-definite. We can use the matrix S to perform a change of variables:

$$\zeta_r = S^{rs} \zeta'_s \quad .$$

Since $|\mathrm{Det}\, S| = 1$, so we obtain for $\mathscr{I}$:

$$\mathscr{I} = e^{-M} \prod_r \int_{-\infty}^{\infty} \mathrm{d}\zeta' \, e^{-\frac{1}{2} k_r \zeta'^2 - (S^{-1}L)^r \zeta'} \quad ,$$

now completing the square in the exponential, and remembering that the measure is translationally invariant, we obtain

$$\mathscr{I} = e^{-M} \prod_r \sqrt{\frac{2\pi}{k_r}} e^{\frac{1}{2k_r}((S^{-1}L)^r)^2} \quad ;$$

and since

$$\det \mathbf{K} = \prod_r k_r \ , \quad K_{rs}^{-1} = S^{rl} S^{sl} k_l^{-1} \ ,$$

$\mathscr{I}$ finally becomes

$$\mathscr{I} = \left(\det \left(\frac{\mathbf{K}}{2\pi} \right) \right)^{-\frac{1}{2}} e^{\frac{1}{2} L^r L^s K_{rs}^{-1} - M} \quad . \tag{C.0.1}$$

If we define $\bar{\zeta}^r = -K_{rs}^{-1} L^s$, we see that

$$\left. \frac{\partial Q}{\partial \zeta^r} \right|_{\zeta = \bar{\zeta}} = 0 \quad ;$$

so we can finally write

$$\mathscr{I} = \left(\det \left(\frac{\mathbf{K}}{2\pi} \right) \right)^{-\frac{1}{2}} e^{-Q(\bar{\zeta})} \quad .$$

These Gaussian integrals can be generalized to the case of functional integrals, substituting $\prod_r \mathrm{d}\zeta^r$ with a functional measure $\mathscr{D}[\zeta(t)]$, considering matrix indices as a condensed multi-index with a discrete and a continuous part, and considering the determinant det as a functional determinant Det. This could be justified by the fact that usually functional measures are translationally invariant and introducing a suitable regularization (such as time-slicing or mode regularization) they effectively reduce to a product $\prod_k \mathrm{d}\zeta^k$. However for path integrals Eq. (C.0.1) is just a formal expression, since we have to give a meaning to the functional determinant Det K, where now K is an operator, and in general there would be additional multiplicative factors due to the definition of functional measure; nevertheless since in perturbative averages global multiplication factors disappear[1], because they are present both in the numerator and denominator of the average, the generalization of (C.0.1) to functional integration is quite useful.

[1] but in general not the functional determinant Det K that could modify the path integral action, as discussed in Section 1.2.3; in particular in curved space functional integration of the momenta causes ambiguities and divergencies in perturbative calculations that lead to the introduction of ghosts and regularization-dependent counterterms.

Part II

CURVED SPACE-TIME

We start this second part introducing a relativistic model to describe the motion of a particle in a curved space-time, and how we can perform a quantization of such model. Starting from the quantum self-adjoint and covariant Hamiltonian operator of our quantized theory we derive path integrals both in phase and configuration space.

After a discussion of ambiguities and divergences we have to face when we make perturbative calculations in configuration space path integral, we introduce ghosts and regularization schemes that solve such divergences and ambiguities by the introduction of counterterms. Such counterterms that are fixed with a two-loop perturbative calculation and depend on the regularization scheme used.

outline of second part

In the last chapter we extend the worldline formalism we introduced in the first part (for the calculation of one-loop effective actions) to the case of a scalar field in a gravitational background. Then we analyze the phase space path integral for the relativistic quantum Hamiltonian operator we already mentioned, and show that in this case is not particularly useful to introduce regularization schemes, at least to perform perturbative calculations, since every calculation is nor ambiguous nor divergent in the continuum limit. Finally we calculate explicitly counterterms for mode regularization in phase space path integral.

Chapter 3

Path Integrals

"Sono ubriaco d'universo"

G.Ungaretti

Differential geometry gives us the tools to describe a particle moving in a curved manifold of space-time. We will develop the usual notions of general relativity to describe a particle moving along its worldline in a space-time manifold characterized by a non-trivial metric tensor. We will then develop some methods to quantize such a particle, and we will give a path integral formulation for its evolution in affine parameters (*e.g.* the proper time for massive particles). We see that the development of a path integral description of particles in curved backgrounds is not trivial: for example it requires regularization schemes and counterterms to treat divergences and ill-defined products of distributions; so we will analyze it in detail. Also, we will see that we can manage the one-loop effective action for a field in a gravitational background with quantum mechanical path integral worldline models.

3.1: Particles in curved manifolds of space-time

We will study now the models to describe a classical free falling particle in a curved space-time, and its quantization.

3.1.1: Relativistic Action

It is not immediate to formulate a variational principle for a test particle of mass m, $m \geq 0$ moving through an arbitrary curved space-time not subjected to any external force (obviously except gravity itself). In fact, we know that such a particle freely falls, *i.e.* it follows a

geodesic motion[25]: such a motion is described by the equation

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha{}_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0 \quad , \tag{3.1.1}$$

where $x^\alpha(\lambda)$ is the coordinate depending on some geodesic affine parameter and the Γs are the metric connection coefficients. This equation does not depend explicitly on mass, so if we want to obtain it from a Lagrangian variational principle $\delta S = 0$ the mass would have to be an overall multiplicative factor, *i.e.* $S = mS'$, but in that case we would have $S \equiv 0$ for massless particles. In fact this is the case even in special relativity, where we define $S = -m \int \mathrm{d}s$. However for massive particles this definition was a wise one: $\mathrm{d}s$ and m are invariant, and from geometric considerations we know that $\mathrm{d}s$ has extremal length on the physical trajectory so the variational principle $\delta S = 0$ is automatically satisfied. The problem is $\mathrm{d}s$ is identically zero for massless particles. However also geodesics of arbitrary space-time manifolds has extremal $\mathrm{d}s = \sqrt{g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu}$, and it is invariant under reparametrization of the affine parameter. So it appears to be a good choice to define the action of a massive particle, but its quantization would result quite difficult, due to the presence of the square root. We must proceed in a different way: we note that if we define a relativistic Hamiltonian function

Geodesic Motion and the choice of Relativistic Action

$$\mathcal{H}(x,p) = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu \quad ;$$

the action

$$S[x,p] = \int (p_\mu \mathrm{d}x^\mu - \mathcal{H}(x,p)\mathrm{d}\lambda) \quad ,$$

where λ is an affine parameter, leads to Eq. (3.1.1) if we use the variational principle

$$\frac{\delta S}{\delta p_\mu} = 0 \quad ,$$
$$\frac{\delta S}{\delta x^\mu} = 0 \quad .$$

This action has three major problems:

1. it is not invariant under reparametrization of the affine parameter λ;
2. it doesn't fully characterize the free-falling particle, since its mass m does not appear;
3. it satisfies the condition that the particle moves along a geodesic, but it allows space-like geodesics as well as time-like or null ones, but the former have to be forbidden for a real particle (we can formulate this condition as the well-known mass-shell condition $g^{\mu\nu} p_\mu p_\nu + m^2 = 0$[1]).

Last two problems are quite natural since we have already discussed that geodesic motion is mass-independent, so it is quite natural that an action could satisfy Eq. (3.1.1) without mass dependence or mass-shell condition.

[1] we use here the signature $(-+++)$ for the flat (Minkowskian) metric $\eta_{\mu\nu}$.

There is an interesting way to solve all these problems, and it is borrowed from Quantum Field Theory. Even if the action S is not invariant under general reparametrization of λ, *i.e.* under local transformations with $\delta\lambda = \varepsilon(\lambda)$, it is invariant under the global transformation $\lambda' = \lambda + \varepsilon$: Taylor-expanding $p(\lambda')$ and $x(\lambda')$ around λ and considering only first order in ε (infinitesimal transformation) we obtain, denoting derivation with respect to λ with a dot,

$$\delta x^\mu = \varepsilon \dot{x}^\mu \quad ,$$
$$\delta p_\mu = \varepsilon \dot{p}_\mu \quad ;$$

so the variation of the action becomes

$$\delta S = \int \mathrm{d}\lambda\, \varepsilon \Big(\dot{p}_\mu \dot{x}^\mu + p_\mu \ddot{x}^\mu - \frac{1}{2} g^{\mu\nu}{}_{,\rho} \dot{x}^\rho p_\mu p_\nu - g^{\mu\nu} \dot{p}_\mu p_\nu \Big) \quad ,$$

for it to be zero for arbitrary ε, it must be zero the quantity inside parentheses. But since

$$\frac{\delta S}{\delta p_\mu} = 0 \Rightarrow \dot{x}^\mu = g^{\mu\nu} p_\nu \quad , \tag{3.1.2}$$

it is straightforward that the first and the last term inside the parentheses cancel each other; then note that Eq. (3.1.1) allows to write the second term as

$$-\frac{1}{2} p_\mu \Gamma^\mu{}_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = -\frac{1}{2} g^{\mu\sigma} p_\mu \Gamma_{\sigma\nu\rho} \dot{x}^\nu \dot{x}^\rho = -\frac{1}{2} (g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}) \dot{x}^\sigma \dot{x}^\nu \dot{x}^\rho = -\frac{1}{2} g_{\sigma\nu,\rho} \dot{x}^\sigma \dot{x}^\nu \dot{x}^\rho \,,$$

where we have used the definition of connection coefficients Γ as derivatives of the metric, the relation (3.1.2), the fact that the metric is symmetric and in the last equality the symmetry of $\dot{x}^\sigma \dot{x}^\nu \dot{x}^\rho$ in the permutation of indices; finally in the third term we use the relation

$$(g_{\mu\nu} g^{\nu\rho})_{,\sigma} = \delta_\mu{}^\rho{}_{,\sigma} = 0 \Rightarrow g^{\mu\nu}{}_{,\rho} = -g^{\mu\sigma} g^{\nu\tau} g_{\sigma\tau,\rho}$$

and relation (3.1.2) to rewrite it as

$$\frac{1}{2} g_{\sigma\nu,\rho} \dot{x}^\sigma \dot{x}^\nu \dot{x}^\rho \quad ,$$

so effectively $\delta S = 0$ for arbitrary ε, thus $\lambda \to \lambda + \varepsilon$ is a global symmetry indeed. It corresponds to the conservation through geodesic motion of $\mathcal{H}(x,p)$, the Nöther current associated with the symmetry. In field theories one introduces gauge fields in order to localize global symmetries, we would like to do the same here: we introduce an auxiliary gauge field $e(\lambda)$[1], usually gauge fields couple with the conserved current associated with the global symmetry, so we will try the same ansatz here, formulating a new action

$$S[x,p,e] = \int \mathrm{d}\lambda\, (p_\mu \dot{x}^\mu - e\mathcal{H}(x,p)) \quad ; \tag{3.1.3}$$

this action is invariant under general reparametrizations $\delta\lambda = \varepsilon(\lambda)$ if the gauge field has a variation under such a transformation $\delta e = 2\dot{\varepsilon}$, so we have constructed an action invariant under general reparametrizations of the worldline, provided that such a transformation

[1] it is usually called the **einbein**.

causes variations

$$\begin{aligned}\delta x^\mu &= \varepsilon \dot{x}^\mu \quad ,\\ \delta p_\mu &= \varepsilon \dot{p}_\mu \quad ,\\ \delta e &= 2\dot{\varepsilon} \quad .\end{aligned}$$

We have not yet analyzed the equations provided in that case by the variational method:

$$\begin{aligned}\frac{\delta S}{\delta e} &= 0 \Rightarrow g^{\mu\nu}p_\mu p_\nu = 0 \quad ,\\ \frac{\delta S}{\delta x^\mu} &= 0 \Rightarrow \dot{p}_\mu = -\frac{e}{2}g^{\nu\rho}{}_{,\mu}p_\nu p_\rho \quad ,\\ \frac{\delta S}{\delta p_\mu} &= 0 \Rightarrow \dot{x}^\mu = e g^{\mu\nu}p_\nu \quad ;\end{aligned}$$

we immediately see that the variational equation for e is the mass-shell condition for a massless particle, however we can derive the equation for geodesic motion from variational equations for x and p only if the einbein is constant. But the introduction of the gauge field introduced also a gauge freedom, underlying that there are more variables than physical degrees of freedom in the system; that freedom allows us to fix the einbein to be constant, so our model effectively describes a massless freely falling particle in a curved space-time. We have thus solved the problems of the action, that now is invariant under worldline reparametrizations and includes the mass-shell condition (in the case of a massless particle); gauge fixing procedure explicit out the dynamics of the system (geodesic equation) restoring physical degrees of freedom, but destroys the covariance of the theory. The easiest way to introduce mass in such framework is to add a fictitious fifth space-time dimension x^5, setting its conjugate momentum $p_5 \equiv m$, and setting

The auxiliary field e makes the action invariant under reparametrizations of the worldline

$$\mathbf{g}(x^\mu, x^5) = \mathbf{g}(x^\mu) = \begin{pmatrix} & & & & 0\\ & & \mathbf{g}(x^\mu) & & 0\\ & & & & 0\\ & & & & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad ;$$

the term $p_5\dot{x}^5 = m\dot{x}^5$ could be dropped, being a total derivative, so the action remains the same as before apart for the additional term $eg^{55}p_5p_5/2 = em^2/2$, finally obtaining an action with the same symmetries as before that reads:

$$S[x,p,e] = \int \mathrm{d}\lambda \left(p_\mu \dot{x}^\mu - \frac{e}{2}(g^{\mu\nu}p_\mu p_\nu + m^2)\right) \quad ; \tag{3.1.4}$$

variational equations now becoming

$$\begin{aligned}\frac{\delta S}{\delta e} &= 0 \Rightarrow g^{\mu\nu}p_\mu p_\nu + m^2 = 0 \quad ,\\ \frac{\delta S}{\delta x^\mu} &= 0 \Rightarrow \dot{p}_\mu = -\frac{e}{2}g^{\nu\rho}{}_{,\mu}p_\nu p_\rho \quad ,\\ \frac{\delta S}{\delta p_\mu} &= 0 \Rightarrow \dot{x}^\mu = e g^{\mu\nu}p_\nu \quad .\end{aligned}$$

Now the equation for e gives the mass-shell condition in the general case of a mass $m \geq 0$, and fixing the gauge freedom with a constant einbein gives the geodesic equation. So we see that the covariant action in Eq. (3.1.4) describes a freely falling particle of mass $m \geq 0$ in curved space-time with metric $g_{\mu\nu}(x)$.

Using the variational equation for p we can get rid of conjugate momenta and obtain the configuration space action

$$S[x,e] = \int \mathrm{d}\lambda \, \frac{1}{2}\Big(e^{-1} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu - e m^2\Big) \quad . \tag{3.1.5}$$

Getting then rid of the einbein e with its variational equation we obtain, for massive theories, $S[x] = -m \int \mathrm{d}\lambda \, \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$.

3.1.2: *Quantization*

Consider the phase space action (3.1.4), remembering the relativistic Hamiltonian $\mathcal{H}(x,p)$ we could write it as:

$$S[x,p,e] = \int \mathrm{d}\lambda \left(p_\mu \dot{x}^\mu - e\Big(\mathcal{H}(x,p) + \frac{m^2}{2}\Big)\right) \quad ;$$

and we see that variational equations could be written as

$$\begin{aligned}
\frac{\delta S}{\delta e} = 0 \Rightarrow \mathcal{H}(x,p) + \frac{m^2}{2} = 0 \quad , \\
\frac{\delta S}{\delta x^\mu} = 0 \Rightarrow \dot{p}_\mu = -e\frac{\partial \mathcal{H}}{\partial x^\mu} \quad , \\
\frac{\delta S}{\delta p_\mu} = 0 \Rightarrow \dot{x}^\mu = e\frac{\partial \mathcal{H}}{\partial p_\mu} \quad ,
\end{aligned}$$

the so-called Hamiltonian equations. Thus defining Poisson brackets in a standard way

$$[f(x,p), g(x,p)]_{p.b.} \equiv \frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial x^\mu} \quad ,$$

we obtain usual canonical relations (apart the presence of the einbein):

$$\begin{aligned}
[x^\mu, p_\nu]_{p.b.} &= \delta^\mu{}_\nu \quad , \\
[x^\mu, e\mathcal{H}(x,p)]_{p.b.} &= \dot{x}^\mu \quad , \\
[p_\mu, e\mathcal{H}(x,p)]_{p.b.} &= -\dot{p}_\mu \quad .
\end{aligned}$$

Since we don't want the einbein in the quantization of the model we use a Gupta-Bleuler like quantization[1]: we fix the gauge freedom, setting $e(\lambda) = 1$, obtaining

$$S[x,p] = \int \mathrm{d}\lambda \, \Big(p_\mu \dot{x}^\mu - \mathcal{H}(x,p)\Big) \quad ,$$

[1] in analogy with the method introduced by Gupta and Bleuler for the quantization of the electromagnetic field.

since the additive factor proportional to m^2 is now a constant unimportant for the equations of motion, however we have to remember the constraint $g^{\mu\nu}p_\mu p_\nu + m^2 = 0$ due to gauge field. Poisson brackets now reduce effectively to the usual ones. We can now canonically quantize the system, defining an Hilbert space $\mathscr{H}$, considering $\hat{x}^\mu$, $\hat{p}^\mu$ and $\mathcal{H}(\hat{x},\hat{p})$ as operators in $L(\mathscr{H})$, so the Poisson brackets become commutators (with an additional i factor[1]), *i.e.*:

$$\begin{aligned}[\hat{x}^\mu, \hat{p}_\nu] &= i\delta^\mu{}_\nu \quad , \\ [\hat{x}^\mu, \mathcal{H}(\hat{x},\hat{p})] &= i\dot{\hat{x}}^\mu \quad , \\ [p_\mu, \mathcal{H}(\hat{x},\hat{p})] &= -i\dot{\hat{p}}_\mu \quad .\end{aligned}$$

As usual $\mathscr{H}$ is isomorphic to a space $L^2(\mathbb{R}^4, \sqrt{-g(x)}\mathrm{d}^4x)$ ($-g(x)$ is the absolute value of the determinant of the metric[2]), where the unitary transform of operator $\hat{x}^\mu$ reduces to multiplication by x^μ and operator $\hat{p}_\mu$ reduces to $-i\partial_\mu$. We have not considered the constraint $\mathcal{H}(x,p) + m^2/2 = 0$ yet. To impose it as an operator constraint, *i.e.* $\mathcal{H}(\hat{x},\hat{p}) + m^2\mathbb{1}/2 = 0$ is not realizable. Accordingly with the Gupta-Bleuler quantization of electrodynamics we will restrict $\mathscr{H}$ to $\mathscr{H}_{ph}$, where

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$$\psi \in \mathscr{H}_{ph} \Leftrightarrow \left(\mathcal{H}(\hat{x},\hat{p}) + \frac{1}{2}m^2\mathbb{1}\right)\psi = 0 \quad . \tag{3.1.6}$$

At this moment we can't avoid any further to analyze the Hamiltonian operator $\mathcal{H}(\hat{x},\hat{p})$. The first ansatz we can try is the direct quantization $\mathcal{H} = g^{\mu\nu}(\hat{x})\hat{p}_\mu\hat{p}_\nu/2$, but it is not even self-adjoint! And self-adjointness is not all, since we want a relativistic theory for a quantum particle valid in any coordinate system, the states of Hilbert space must be invariant under general coordinate transformation, and if we impose that condition at a fixed value of the affine parameter, it must hold at any value: but since evolution is dictated by the Hamiltonian function (Schrödinger equation), it has to be also generally covariant. So we are looking for a self-adjoint Hamiltonian operator invariant under general coordinate transformations[26].

Theorem 3.1.1. *The self-adjoint Hamiltonian operator*

$$\mathcal{H}(\hat{x},\hat{p}) = \frac{1}{2}(-g(\hat{x}))^{-\frac{1}{4}}\hat{p}_\mu(-g(\hat{x}))^{\frac{1}{2}}g^{\mu\nu}(\hat{x})\hat{p}_\nu(-g(\hat{x}))^{-\frac{1}{4}} \tag{3.1.7}$$

is invariant under general coordinate transformations and classically equivalent to

$$\mathcal{H}(x,p) = \frac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu \quad .$$

Proof. Classically an infinitesimal general transformation is of the form $x'^\mu = x^\mu + \xi^\mu(x)$, *i.e.* $\delta x^\mu = \xi^\mu(x)$, where $\xi^\mu(x)$ is an infinitesimal arbitrary function. The ps, being covariant, would transform as

$$p'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}p_\nu \quad ,$$

[1] we have set $\hbar = 1$.

[2] $\sqrt{-g(x)}\mathrm{d}^4x$ is the invariant measure in arbitrary curved space-time.

i.e. $\delta p_\mu = -\partial_\mu \xi^\nu p_\nu$. Performing two consecutive transformations we find

$$[\delta_1, \delta_2] x^\mu = \xi_2^{nu}(x)\partial_\nu \xi_1^\mu(x) - \xi_1^\nu(x)\partial_\nu \xi_2^\mu(x) \quad ;$$

so the generator $\mathscr{G}$ of quantum transformations has to satisfy the algebra

$$[\mathscr{G}(\xi_1), \mathscr{G}(\xi_2)] = \mathscr{G}(\xi_2^\mu \partial_\mu \xi_1 - \xi_1^\mu \partial_\mu \xi_2) \quad .$$

It is possible to see that $\mathscr{G}$ splits into an orbital and a "spin" part satisfying the following relations:

$$\begin{aligned}
\mathscr{G} &= \mathscr{G}_o + \mathscr{G}_s \quad , \quad \text{with} \\
\mathscr{G}_o &= -\frac{i}{2}(\xi^\mu p_\mu + p_\mu \xi^\mu) \quad , \\
\mathscr{G}_s &= \int \mathrm{d}^4 x \left(\xi^\rho g_{\mu\nu,\rho} + (\partial_\mu \xi^\rho) g_{\rho\nu} + (\partial_\nu \xi^\rho) g_{\rho\mu} \right) \frac{\delta}{\delta g_{\mu\nu}} \quad .
\end{aligned}$$

In fact, calculating commutators with $\mathscr{G}$ we obtain the right quantum transformations (from now on we will omit $\hat{}$ over x and p since we deal only with operators, so confusion is avoided):

We choose the Quantum Hamiltonian to be covariant and self-adjoint

$$\begin{aligned}
\delta x^\mu &= [x^\mu, \mathscr{G}] = \xi^\mu(x) \quad , \\
\delta p_\mu &= [p_\mu, \mathscr{G}] = -\frac{1}{2}(p_\nu \partial_\mu \xi^\nu + \partial_\mu \xi^\nu p_\nu) \quad , \\
\delta g_{\mu\nu} &= [g_{\mu\nu}, \mathscr{G}] = -(\partial_\mu \xi^\rho) g_{\rho\nu} - (\partial_\nu \xi^\rho) g_{\rho\mu} \quad ;
\end{aligned}$$

where the last transformation is the infinitesimal counterpart of the finite transformation rule

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \quad ,$$

so the determinant transforms as

$$\begin{aligned}
-g'(x') &= -\left|\frac{\partial x}{\partial x'}\right|^2 g(x) \quad , \quad \text{with} \\
\left|\frac{\partial x}{\partial x'}\right| &= \det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right) \quad .
\end{aligned}$$

Since infinitesimally

$$\frac{\partial x^\mu}{\partial x'^\nu} = \delta^\mu_\nu - \partial_\nu \xi^\mu \quad ,$$

we can generalize momentum infinitesimal transformation rule to a finite one in that way:

$$p'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} p_\nu - \frac{1}{2}\left[\frac{\partial x^\nu}{\partial x'^\mu}, p_\nu\right] \quad ;$$

but since $[f(x), p_\mu] = i\partial_\mu f(x)$ we obtain

$$\left[\frac{\partial x^\nu}{\partial x'^\mu}, p_\nu\right] = i \frac{\partial x'^\rho}{\partial x^\nu} \frac{\partial^2 x^\nu}{\partial x'^\rho \partial x'^\mu} \quad ,$$

so finally remembering that $\ln\det\mathbf{A} = \mathrm{tr}\ln\mathbf{A}$ we have

$$p'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\left(p_\nu - \frac{i}{2}\partial_\nu \ln\left|\frac{\partial x}{\partial x'}\right|\right) \quad .$$

Using all these relations we see that

$$(-g')^{\frac{1}{4}} p'_\mu (-g')^{-\frac{1}{4}} = \frac{\partial x^\nu}{\partial x'^\mu}(-g)^{\frac{1}{4}} p_\nu (-g)^{-\frac{1}{4}} \quad ,$$
$$(-g')^{-\frac{1}{4}} p'_\mu (-g')^{\frac{1}{4}} = (-g)^{-\frac{1}{4}} p_\nu (-g)^{\frac{1}{4}} \frac{\partial x^\nu}{\partial x'^\mu} \quad ,$$

so since

$$g'^{\mu\nu}(x') = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} g^{\rho\sigma}(x) \quad ,$$

we see that $\mathcal{H}(x',p') = \mathcal{H}(x,p)$. □

Now that we know the right quantum form of covariant Hamiltonian operator we would like to see its action on $L^2(\mathbb{R}^4, \sqrt{-g(x)}\mathrm{d}^4x)$: since in non-relativistic framework it is proportional to Laplacian operator, in our relativistic case we would expect it to be proportional to the covariant Laplacian operator. It is indeed the case: if $f(x) \in L^2(\mathbb{R}^4, \sqrt{-g(x)}\mathrm{d}^4x)$ we know that

$$(\mathcal{H}(x,\partial_x)f)(x) = -\frac{1}{2}(-g(x))^{-\frac{1}{4}}\partial_\mu\left((-g(x))^{\frac{1}{2}} g^{\mu\nu}(x)\partial_\nu\left((-g(x))^{-\frac{1}{4}} f(x)\right)\right) \quad ;$$

the problem is to calculate the derivative of $(-g(x))^a$, where $a = 1/2, -1/4, \ldots$; but

$$\partial_\alpha h(x)^a = a h(x)^a \partial_\alpha \ln h(x)$$

if $a < 1$, so using the property $\partial_\alpha \ln|\det\mathbf{A}| = \mathrm{tr}(\mathbf{A}^{-1}\partial_\alpha\mathbf{A})$ valid for any matrix $\mathbf{A}$, we have

$$\partial_\mu(-g)^a = a(-g)^a g^{\nu\rho} g_{\nu\rho,\mu} \quad ;$$

performing all the boring calculations and finally remembering that if ∇_μ is the covariant derivative, acting on a scalar function like $f(x)$ we have $\nabla_\mu f(x) \equiv \partial_\mu f(x)$, we obtain indeed

$$\mathcal{H}f = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla_\nu f \equiv -\frac{1}{2}\nabla^2 f \quad ,$$

i.e. the Hamiltonian operator acts like minus the covariant Laplacian times one-half, as we expected.

Now it is clear what is the meaning of the mass-shell constraint $(\mathcal{H} + m^2/2)f = 0$, that restrict the Hilbert space to the physical one: it could be written as

$$(-\nabla^2 + m^2)f(x) = 0$$

i.e. physical states $f \in \mathscr{H}_{ph}$ must obey generalized Klein-Gordon equation. Conversely, we can suggest that imposing Klein-Gordon equation on a particle is analogous to constraining it on the mass-shell, and this is indeed a necessary condition to avoid non-causal, unphysical behaviors.

3.2: *Derivation of path integral*

Covariant Hamiltonian describes evolution in λ, with Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi = \mathcal{H}(x,\nabla_x)\Psi \quad ,$$

So if the Hamiltonian is explicitly λ-independent[1] the unitary operator that describes evolution in λ is

$$\mathcal{U}(\lambda_f - \lambda_i) = e^{-i(\lambda_f-\lambda_i)\mathcal{H}} \quad ;$$

so we would like to calculate the integral kernel $\mathcal{K}(x_f,\lambda_f;x_i,\lambda_i)$, that satisfies equation

$$\psi(x_f,\lambda_f) = \int \mathrm{d}^4x\,\sqrt{-g(x_i)}\,\mathcal{K}(x_f,\lambda_f;x_i,\lambda_i)\psi(x_i,\lambda_i) \quad ,$$

in a path integral form. We derive this result in a more general setting: we extend the number of space-time dimensions to a value D, and we perform a Wick rotation both in λ and in the target time coordinates (in D space-time dimensions we could have more than one time coordinate), so we are considering an Euclidean D-dimensional space with a metric[2] $g_{ij}(x)$ that locally reduces to δ_{ij}. We call the rotated affine parameter t, so previous formulas would read[3]

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\Psi &= \mathcal{H}(x,\nabla_x)\Psi \quad , \\ \mathcal{U}(t_f - t_i) &= e^{-(t_f-t_i)\mathcal{H}} \quad , \\ \psi(x_f,t_f) &= \int \mathrm{d}^Dx\,\sqrt{g(x_i)}\,\mathcal{K}(x_f,t_f;x_i,t_i)\psi(x_i,t_i) \quad . \end{aligned}$$

All the considerations about the covariant Hamiltonian of previous section extend immediately to Euclidean affine evolution in Euclidean target space-time. We will see that to calculate effective actions it is quite useful to extend the Hamiltonian introducing an external space-time dependent scalar potential $V(x)$, so we are considering

$$\mathcal{H}(\hat{x},\hat{p}) = \frac{1}{2}g(\hat{x})^{-\frac{1}{4}}\hat{p}_i g(\hat{x})^{\frac{1}{2}}g^{ij}(\hat{x})\hat{p}_j g(\hat{x})^{-\frac{1}{4}} + V(\hat{x}) \quad ;$$

and since $V(x)$ is a scalar function the covariance of the Hamiltonian is not affected.

3.2.1: *Time slicing*

As we saw in Part I, the integral kernel $\mathcal{K}$ reduces to the calculation of the transition element

$$\mathcal{K}(x_f,t_f;x_i,t_i) = \mathcal{K}(x_f,\beta;x_i,0) = \langle x_f|e^{-\beta\mathcal{H}}|x_i\rangle \quad ,$$

[1] and therefore it is also implicitly independent, since ∂_λ *mathcalH* $= \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{H}$.

[2] we will use roman indices to indicate we are working in a space-time locally Euclidean.

[3] since the metric is now positive definite we avoid the minus sign in the absolute value of the determinant.

where as usual $\beta \equiv t_f - t_i$. Since as we mentioned in the previous section the Hilbert space we are considering is isomorphic to $L^2(\mathbb{R}^D, \sqrt{g(x)}\mathrm{d}^D x)$, we need to use the following normalizations for position and momentum eigenstates:

$$\mathbb{1} = \int \mathrm{d}^D x \sqrt{g(x)} |x\rangle \langle x| \quad \Rightarrow \quad \langle y|x\rangle = \frac{\delta^D(x-y)}{\sqrt{g(x)}} \quad ,$$
$$\mathbb{1} = \int \mathrm{d}^D p \, |p\rangle \langle p| \quad \Rightarrow \quad \langle p'|p\rangle = \delta^D(p-p') \quad ,$$

so the $\langle x|p\rangle$ element becomes

$$\langle x|p\rangle = \left((2\pi)^{\frac{D}{2}} g(x)^{-\frac{1}{4}}\right) e^{ip_i x^i} \quad .$$

We can now proceed as in Section 1.2, setting $\beta = N\epsilon$, and we obtain:

$$\mathcal{K}(x_f, \beta; x_i, 0) = \lim_{\epsilon \to 0} \int \prod_{k=1}^{N} \mathrm{d}^D p_k \prod_{k=1}^{N-1} \mathrm{d}^D x_k \langle x_f | e^{-\epsilon \mathcal{H}} | p_N \rangle \sqrt{g(x_{N-1})} \cdots \langle x_1 | e^{-\epsilon \mathcal{H}} | p_1 \rangle \langle p_1 | x_i \rangle \, . \tag{3.2.1}$$

The problem is to evaluate the transition element

$$\langle x_k | e^{-\epsilon \mathcal{H}} | p_k \rangle \quad ,$$

Weyl ordering

since $\mathcal{H}$ is not in normal ordering. However instead of rewriting it in normal ordered form, we will use a different ordering, the **Weyl ordering**[1], since a theorem proved in Appendix D states that if we have an operator $O_W(\hat{x}, \hat{p})$ in Weyl ordering

$$\int \mathrm{d}^D p_k \langle x_k | O_W(\hat{x}, \hat{p}) | p_k \rangle \langle p_k | x_{k-1} \rangle = \int \mathrm{d}^D p_k O_W(\bar{x}_{k-1/2}, p_k) \langle x_k | p_k \rangle \langle p_k | x_{k-1} \rangle \quad ,$$

where $\bar{x}_{k-1/2} = (x_k + x_{k-1})/2$. In Eq. (3.2.1) we could use this theorem to evaluate:

$$\int \mathrm{d}^D p_k \langle x_k | \left(e^{-\epsilon \mathcal{H}}\right)_W | p_k \rangle \langle p_k | x_{k-1} \rangle = \int \mathrm{d}^D p_k \left(e^{-\epsilon \mathcal{H}(\bar{x}_{k-1/2}, p_k)}\right)_W \langle x_k | p_k \rangle \langle p_k | x_{k-1} \rangle \quad ,$$

however if it might be difficult to write $\mathcal{H}$ in a closed Weyl ordered form, it is almost impossible to write $\exp\{-\epsilon\mathcal{H}\}$ in Weyl ordering. However, as discussed in Appendix D, we can make the approximation, at first order in ϵ (justified since we are taking the limit $\epsilon \to 0$):

$$\left(e^{-\epsilon \mathcal{H}}\right)_W = e^{-\epsilon \mathcal{H}_W} \quad .$$

[1] We say that, given an operator $O(x,p)$ (we omit hats over x and p, but we are dealing with operators), its Weyl ordered form $O_W(x,p)$ is the same operator where all the xs and ps are arranged in a symmetrical form, plus terms due to $[x,p]$. As an example consider the operator xp [26]:

$$xp = \frac{1}{2}(xp+px) + \frac{1}{2}(xp-px) = (xp)_S + \frac{1}{2}[x,p] = (xp)_S + \frac{1}{2}i = (xp)_W \quad ;$$

where we have called $(xp)_S$ the symmetrical part $(xp+px)$ of xp.

We also note that it could be quite difficult to write a symmetric form for $\mathcal{H}$ if it contains an additive, only x-dependent term as $V(x)$. So in the evaluation of

$$\langle x_k|e^{-\epsilon\mathcal{H}}|p_k\rangle \quad ,$$

we use Trotter formula to write it as

$$\langle x_k|e^{-\epsilon\mathcal{H}_0}e^{-\epsilon V}|p_k\rangle = e^{-\epsilon V(x_k)}\langle x_k|e^{-\epsilon\mathcal{H}_0}|p_k\rangle \quad ,$$

with

$$\mathcal{H}_0(\hat{x},\hat{p}) = \frac{1}{2}g(\hat{x})^{-\frac{1}{4}}\hat{p}_i g(\hat{x})^{\frac{1}{2}}g^{ij}(\hat{x})\hat{p}_j g(\hat{x})^{-\frac{1}{4}} \quad .$$

The problem is that $\mathcal{H}_{0,W}$ is evaluated at the midpoint $\bar{x}_{k-1/2}$, while V is evaluated at x_k. That is not a problem, since we can show that at order zero in ϵ, $V(x_k) = V(\bar{x}_{k-1/2})$[1]: expanding $V(x_k)$ around $\bar{x}_{k-1/2}$ we obtain

$$V(x_K) = V(\bar{x}_{k-1/2}) + \sum_{n=1}^{\infty}\frac{V^{(n)}(\bar{x}_{k-1/2})}{n!}\left(\frac{x_k - x_{k-1}}{2}\right)^n \quad ;$$

but $x_k - x_{k-1} \equiv x(t_{k-1}+\epsilon) - x(t_{k-1})$, so expanding $x(t_{k-1}+\epsilon)$ about $x(t_{k-1})$ we obtain

$$V(x_K) = V(\bar{x}_{k-1/2}) + \sum_{n=1}^{\infty}\frac{V^{(n)}(\bar{x}_{k-1/2})}{n!}\left(\frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{j!}x^{(j)}(t_{k-1})\epsilon^j\right)^n \quad ;$$

so we have shown that at zeroth order in ϵ the relation is satisfied. From now on when we write $\mathcal{H}_W(\bar{x}_{k-1/2},p_k)$ we mean

$$\mathcal{H}_W(\bar{x}_{k-1/2},p_k) = \mathcal{H}_{0,W}(\bar{x}_{k-1/2},p_k) + V(\bar{x}_{k-1/2}) \quad .$$

Some effort and a lot more calculation permits to find $\mathcal{H}_{0,W}$:

$$\mathcal{H}_{0,W}(\hat{x},\hat{p}) = \left(\frac{1}{2}g^{ij}(\hat{x})\hat{p}_i\hat{p}_j\right)_S + \frac{1}{8}(R(\hat{x}) + g^{ij}(\hat{x})\Gamma^k{}_{il}(\hat{x})\Gamma^l{}_{jk}(\hat{x})) \quad ;$$

where $R(x)$ is the scalar curvature, and $\Gamma(x)$ the metric connection coefficients.

We can summarize our results, calculating the integral kernel $\mathcal{K}$ as:

$$\mathcal{K}(x_f,t_f;x_i,t_i) = \lim_{\epsilon\to 0}(g(x_f)g(x_i))^{-\frac{1}{4}}\int\prod_{k=1}^{N}\frac{\mathrm{d}^D p_k}{(2\pi)^D}\prod_{k=1}^{N-1}\mathrm{d}^D x_k\, e^{-S_N} \quad , \tag{3.2.2}$$

with

$$S_N = \epsilon\sum_{k=1}^{N}\left[\mathcal{H}_W(\bar{x}_{k-1/2},p_k) - ip_k\cdot\frac{(x_k - x_{k-1})}{\epsilon}\right] \quad ,$$

[1] we stop to order zero because V multiplies ϵ, and since our entire approximation is at first order in ϵ we can approximate V at order zero.

where we identify $x_N \equiv x_f$ and $x_0 \equiv x_i$. Taking the formal limit we obtain the phase space path integral formula:

$$\mathcal{K}(x_f,\beta;x_i,0) = \int\limits_{\substack{x(0)=x_i\\ x(\beta)=x_f}} \mathscr{D}[p(t)]\mathscr{D}[x(t)]\; e^{-S_E[x,p,\beta]} \quad , \tag{3.2.3}$$

where the phase space Euclidean action is

$$S_E[x,p,\beta] = \int_0^\beta \mathrm{d}t\,(-ip\cdot\dot{x}+\mathcal{H}_W(x,p)) = \int_0^\beta \mathrm{d}t\left(-ip\cdot\dot{x}+\frac{1}{2}g^{ij}(x)p_ip_j+V(x)+V_{TS}(x)\right) \quad ,$$

where we have called the time-slicing counterterm V_{TS} the Weyl ordering term

$$V_{TS}(x) = \frac{1}{8}(R+g^{ij}\Gamma^k{}_{il}\Gamma^l{}_{jk}) \quad .$$

Path integral in curved space needs counterterms dependent on the regularization scheme chosen

In that case the meaning of the functional measure $\mathscr{D}[x(t)]\mathscr{D}[p(t)]$ is not clear, so the formal expression (3.2.3) is meaningless unless we introduce a proper regularization scheme to manage the otherwise ambiguous functional integration: in particular we will see that constructing a perturbative theory in continuum limit leads to divergences and undefined products of distributions that assume significance only in a regulated environment. Time slicing procedure[27, 28] that led to (3.2.3) is one example of such regularization schemes. At the discretized level everything is well defined and the perturbative theory produces unambiguous results. As we will see in next sections, we can say that $\mathcal{K}$ has a well defined path integral provided we introduce a regularization scheme, and add a regularization-dependent counterterm $\int \mathrm{d}t\, V_{CT}$ to the classical Euclidean action. However, as we will discuss, this model is "super-renormalizable", so a single counterterm is sufficient. Another regularization-dependent factor, called a global renormalization factor, arise in order to properly define the path integral functional measure, however it is fixed by consistency conditions[1].

Therefore every time we write the path integral formulation

$$\mathcal{K}(x_f,t_f;x_i,t_i) = \int\limits_{\substack{x(0)=x_i\\ x(\beta)=x_f}} \mathscr{D}[p(t)]\mathscr{D}[x(t)]\; e^{-S_E[x,p,\beta]} \quad , \tag{3.2.4}$$

we must keep in mind we have to fix a regularization scheme in order to solve ambiguities, and therefore the action reads

$$S_E[x,p,\beta] = \int_0^\beta \mathrm{d}t\left(-ip_i\dot{x}^i+\frac{1}{2}g^{ij}p_ip_j+V(x)+V_{CT,ph}(x)\right) \quad ;$$

where the counterterm depends on the regularization scheme we have chosen, and on the fact we are dealing with a phase space formulation. In the case of time slicing we have seen that the counterterm $V_{TS,ph}(x) = (R+g^{ij}\Gamma^k{}_{il}\Gamma^l{}_{jk})/8$.

[1] as an example by the condition that

$$\psi(x_f,t_f) = \int \mathrm{d}^D x_i\,\sqrt{g(x_i)}\mathcal{K}(x_f,t_f;x_i,t_i)\psi(x_i,t_i)$$

satisfies the Schrödinger equation.

3.2.2: *Lagrangian formulation*

Although we have to be careful when we deal with the continuum expression (3.2.4), we can extend some considerations we have done in flat space-time. In particular we would like to integrate out the momenta, in order to formulate a Lagrangian version of path integral. In Section 1.2.3 we saw that it is possible for a wide range of Hamiltonian functions, provided such function is a quadratic form of the momenta. Well it is indeed the case, so we can repeat the procedure described in that section and integrate out the momenta[1]: at a first glance the price to pay is, apart some constant factors that are easily adsorbed in the functional measure $\mathscr{D}[x(t)]$, a functional determinant that now, unlike most flat-space cases, is $x(t)$-dependent. In particular it is the determinant of the inverse of the metric to the power of minus one-half, but since the determinant of the inverse is the inverse of the determinant we obtain a factor $\sqrt{g(x(t))}$. So we have to deal with the formula

$$\mathcal{K}(x_f, t_f; x_i, t_i) = \int\limits_{\substack{x(0)=x_i \\ x(\beta)=x_f}} \mathscr{D}[x(t)] \sqrt{g(x(t))}\, e^{-S_E[x,\beta]} \quad ,$$

where the Euclidean action is

$$S_E[x,\beta] = \int_0^\beta \mathrm{d}t \left(\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + V(x) + V_{CT,conf}(x) \right) \quad .$$

Integrating out momenta is a tricky procedure: ambiguities and divergencies arise

First of all we expect that the formal functional integration over momenta is not able to preserve the regularization introduced in phase space, so the counterterm $V_{CT,conf}(x)$ would differ from the phase space one, even in the same regularization environment. Furthermore the additional $\sqrt{g(x(t))}$ could be seen as the factor needed to make the functional measure covariant, by generalization of the usual vector space measure: in fact while the measure element in phase space $\mathrm{d}^D p \mathrm{d}^D x$ is relativistically covariant, in configuration space the covariant measure element is $\mathrm{d}^D x \sqrt{g(x)}$.

Since as we will see shortly we are able to manage that functional determinant in a quite elegant way, it seems that integrating out the momenta was not a big deal, unfortunately it is not the case: in fact through a careful analysis of phase space path integrals, as we will do in the next chapter, emerges an interesting fact, *i.e.* that the phase space model does not present divergences or ambiguities, and perturbative calculations could be done directly in the continuum limit. So the price to pay when we pass to configuration space is higher than what it seems: in getting rid of the momenta we introduce divergences and ambiguities in the continuum limit and thus the need of regularization schemes and suitable counterterms for perturbative calculations. However a more detailed analysis and further considerations will be done in the next chapter, by now we focus on the analysis of configuration space path integral.

[1] the calculations in Section 1.2.3 are in real time, however the generalization to Euclidean case is straightforward and doesn't present any theoretical difficulty.

3.3: Continuum Manipulations

We proceed with our analysis of configuration phase path integral for a quantum mechanical model in curved space-time developing some general concepts in the continuum limit. However we recall that such manipulations are only formal, they are justified and meaningful only when we introduce a regularization scheme.

3.3.1: Ghosts

Even if we could identify $\mathscr{D}[x(t)]\sqrt{g(x(t))}$ as the properly covariant functional measure with proper regularization, surely it is not translationally invariant. Since in order to calculate free propagators and perform perturbative calculations we would like to use generating functionals and functional derivatives as we did in Section 1.3.3, we need the measure to be translationally invariant in order to perform Gaussian integrals. We need some trick to rewrite the factor $\sqrt{g(x(t))}$, and restore translational invariance. But since such factor is a functional determinant, we can rewrite it as a functional integral in some auxiliary fields, that will be called "ghosts", since they don't appear in our original model, but as we will see their diagrams are fundamental to eliminate the divergences of the model. We see again that the transition from phase space to configuration space is quite awkward: integrating out momenta we recover Lagrangian formulation of the theory, but introduce otherwise non present divergences, and break translational invariance of the measure; in order to set up things we need to introduce some auxiliary fields (three, as we will see) otherwise unpredicted in our model.

Introduction of Ghost Auxiliary Fields to exponentiate the metric determinant

We know very well that Gaussian path integrals of a real function

$$\int \mathscr{D}[a(t)] \exp\left\{-\frac{1}{2}\int \mathrm{d}t'\mathrm{d}t''\, A_{ij}(t',t'')a^i(t')a^j(t')\right\} = (\mathrm{Det}\, A)^{-\frac{1}{2}} \quad ,$$

if the functional measure is properly defined[1]. But in our case we have $g^{1/2}$, so a real field $a(t)$ is not sufficient. However, Gaussian integration of two Grassmann functions $b(t)$ and $c(t)$ defined as

$$\int \mathscr{D}[b(t)]\mathscr{D}[c(t)] \exp\left\{-\int \mathrm{d}t'\mathrm{d}t''\, A_{ij}(t',t'')b^i(t')c^j(t')\right\} = \mathrm{Det}\, A \quad ,$$

so we can rewrite

$$\sqrt{g(x(t))} \sim \int \mathscr{D}[a(t)]\mathscr{D}[b(t)]\mathscr{D}[c(t)]\; e^{-S_{gh}[a,b,c,\beta]} \quad ,$$

where $a(t)$ is a real function, $b(t)$ and $c(t)$ Grassmann functions and

$$S_{gh}[a,b,c,\beta] = \int_0^\beta \mathrm{d}t\, \frac{1}{2} g_{ij}(x)\left(a^i a^j + b^i c^j\right) \quad ,$$

[1] in general the result is proportional to $(\mathrm{Det}\, A)^{-1/2}$, however the constant factor can be adsorbed or in the definition of the ghost functional measure or in the global renormalization constant of the regularization scheme.

and the $\sim$ symbol means that the eventual constant factor resulting from integrating ghosts is adsorbed in the global renormalization constant of the chosen regularization scheme.

We can summarize our results writing:

$$\mathcal{K}(x_f, t_f; x_i, t_i) = \int\limits_{\substack{x(0)=x_i \\ x(\beta)=x_f}} \mathscr{D}[x(t)]\mathscr{D}[a(t)]\mathscr{D}[b(t)]\mathscr{D}[c(t)]\; e^{-S_E[x,a,b,c,\beta]} \quad , \tag{3.3.1}$$

where the total action $S_E[x, a, b, c, \beta]$ is:

$$S_E[x, a, b, c, \beta] = S_E + S_{gh} = \int_0^\beta \mathrm{d}t \left(\frac{1}{2} g_{ij}(\dot{x}^i \dot{x}^j + a^i a^j + b^i c^j) + V(x) + V_{CT}(x) \right) \quad .$$

3.3.2: *Analysis of divergences*

We have already mentioned that since this theory is super-renormalizable a single counterterm is sufficient to give correct results at any perturbative order[26]: we will now justify our assertion analyzing our model with power counting techniques borrowed from quantum field theory.

The basic assumption we make is that we can see the path $x(t)$ as a classical field of the single time variable t, or in QFT terminology, that we are dealing with a $(0+1)$-dimensional quantum field theory[1]: so if we call the usual field $\varphi^n(x^\mu)$, we identify x^μ with the one dimensional $t \in [0, \beta]$, and φ^n with x^i. We see that in that case the 1-dimensional "space-time" t is finite, so the Fourier expansion of the D massless[2] scalar fields x^i will be a series that takes into account properly the boundary conditions of the xs at $t = 0, \beta$. Since, as discussed in the next section, we will usually shift the field to $q^i(t)$ in a way that q vanishes at the boundaries, the proper Fourier series will be a sine series, and the free propagators become proportional to

$$\Delta(t, t') = -\sum_{m=1}^{\infty} \frac{2}{\pi^2 m^2} \sin(\pi m t) \sin(\pi m t') \quad ,$$

as we will see in detail. So in the momentum space Feynman rules any internal line would be associated to a discrete sum rather than to an integral over "momenta". Furthermore since t is finite, exists a minimal frequency[3]: if we factorize eventual zero modes (modes with null frequency), we don't have any infrared divergency, caused by $p \to 0$ in massless QFT theories, since minimal frequency is fixed and $\neq 0$.

We have ultraviolet problems instead: they are due to arbitrarily great momenta, and in the limit $m \to \infty$ the sum of $\frac{1}{m^2}$ is equivalent to an integral over $\frac{1}{k^2}$. Physically, this is due to the fact that ultraviolet divergencies should not feel the boundaries, since they are related to the microscopical structure of the theory. So in our discussion of ultraviolet divergencies we

[1] we write $0+1$ since we have no spatial dimensions and one temporal dimension, however since we are in Euclidean time this distinction is quite meaningless.

[2] a massive term would be determined by the coupling constant of a factor $\propto x^2$ in the action, but unless it appears in the potential $V(x)$, and we will see that usually it doesn't, such field theory is a massless one.

[3] *i.e.* momentum, since in that model we identify momentum $p \sim \pi m$, the latter being indeed a frequency.

will consider Feynman graphs with usual integration rules and propagators proportional to $\frac{1}{k^2}$, obviously in momentum space.

The action we have to analyze is

$$S = \int \mathrm{d}t \left(\frac{1}{2} g(q) \dot{q}\dot{q} + V(q) \right) \quad ,$$

corresponding to our model, omitting for simplicity vectorial indices. Since we have $\hbar = c = 1$ the action is a pure number, time has the dimension of the inverse of a mass: $[t] = M^{-1}$; and obviously $[q] = M^{-1/2}$. If we expand the metric and the potential in a Taylor series we have:

$$V(q) = \sum_{n=0}^{\infty} V_n q^n \ , \quad g(q) = \sum_{n=0}^{\infty} g_{n+2}\, q^n \ ,$$

where the subscript indicates how many fields a given vertex contains. So easily we deduce that

$$[V_n] = M^{\frac{n}{2}+1} \ ; \quad [g_n] = M^{\frac{n}{2}-1} \ .$$

Interaction corresponds to terms with $n \geq 3$, so all coupling constants have positive mass dimensions. We know that such a theory in QFT is called super-renormalizable: namely, from a certain loop level on there are no more superficial divergencies. Furthermore we will see that divergencies are canceled by ghosts, even if in an ambiguous way, so renormalization involves only finite terms: the $V_{CT}(x)$ already introduced.

The model is Super-Renormalizable

Let's develop our considerations in detail: as usual in QFT we say that a diagram has a superficial degree of divergency D if it corresponds to an integral

$$\int^{\infty} \mathrm{d}k\, k^{D-1} \quad .$$

Obviously $D > 0$ corresponds to a behavior $\sim k^D$ near infinity; $D = 0$ corresponds to $\sim \ln k$, and we have superficial convergence for $D < 0$. Since any propagator behaves like $\frac{1}{k^2}$, each internal line in the diagram provides a factor -2 to D. Any loop corresponds to $\int \mathrm{d}k$, so to a factor $+1$ to D. Any interaction term with field derivatives also corresponds to an additional factor $+1$ to D, as in the case of g interaction terms that contain two derivatives. To summarize, if we call L the number of loops in the diagram, I the number of internal lines, and g_n the number of vertices with that coupling constant in the diagram considered, its maximum degree of divergency would be

$$D = L - 2I + 2\sum_n g_n \quad .$$

Since for a connected diagram holds the following topological identity:

$$L = I - \sum_n (V_n + g_n) + 1 \quad ,$$

we obtain

$$D = 2 - L - 2\sum_n V_n \quad ;$$

and it is clear that such formula makes sense only for $L > 0$, since at the tree level no integrations survive after we impose momentum conservation. We see immediately that any diagram with one or more V vertex converges; while if $V_n = 0 \quad \forall n$, we can have superficially divergent diagrams only when $L \leq 2$, as expected in a super-renormalizable theory.

Divergencies are eliminated by the divergent contributions of ghosts: any interaction $g_n q^n \dot{q}\dot{q}$ is balanced by a term $g_n q^n (aa + bc)$. The remaining logarithmic divergencies usually could be eliminated with symmetric integration, however finite results remain ambiguous: they depend on the regularization scheme we are considering. In particular Feynman graphs contain products of distributions, that are not define mathematically, and this is the ambiguity that has to be solved at the regularized level.

3.3.3: *Perturbative Expansions*

As we did in flat space-time, we would like to find free propagators, in order to perform perturbative calculations. In order to do that we have to split the action in Eq. (3.3.1) into a free and an interacting part. Furthermore, we need a controlling parameter for the perturbative series, and we choose β, so we single it out from the action doing the reparametrization $\tau = t/\beta - 1$, and suitably redefining ghosts obtaining[1]:

$$\frac{1}{\beta} S[x,a,b,c] = \frac{1}{\beta}\int_{-1}^{0} \mathrm{d}\tau \left(\frac{1}{2} g_{ij}(x)(\dot{x}^i\dot{x}^j + a^i a^j + b^i c^j) + \beta^2 \big(V(x) + V_{CT}(x)\big)\right) \quad .$$

In order to do perturbative calculations, we have to split $S = S_2 + S_{int}$, where the free part is used to find propagators and thus has to be quadratic in coordinates and ghosts. In particular we find:

$$S_2 = \int_{-1}^{0} \mathrm{d}\tau \left(\frac{1}{2} g_{ij}(x_f)(\dot{x}^i\dot{x}^j + a^i a^j + b^i c^j)\right) \quad , \tag{3.3.2}$$

$$S_{int} = \int_{-1}^{0} \mathrm{d}\tau \left(\frac{1}{2} [g_{ij}(x) - g_{ij}(x_f)](\dot{x}^i\dot{x}^j + a^i a^j + b^i c^j) + \beta^2 (V(x) + V_{CT}(x))\right) \quad ; \tag{3.3.3}$$

where $g(x_f)$ is the metric calculated at the final point and is therefore a constant.

It is easier to interpret the meaning of propagators if we have vanishing boundary conditions, so we make the translation $q^i(\tau) \equiv x^i(\tau) - x^i_f + \xi^i \tau$, where $\xi^i \equiv x^i_i - x^i_f$; we see immediately that $q(0) = q(-1) = 0$, and since our functional measure $\mathscr{D}[x(\tau)]$ is transla-

[1] ghosts redefinition will be adsorbed in the definition of their functional measure, *i.e.* fixed be the global renormalization constant of the model.

tionally invariant, we obtain:

$$\begin{aligned}
\mathcal{K}(x_f,t_f;x_i,t_i) &= e^{-\frac{1}{2\beta}g_{ij}(x_f)\xi^i\xi^j}\int\limits_{VBC}\mathscr{D}[q(\tau)]\mathscr{D}[a(\tau)]\mathscr{D}[b(\tau)]\mathscr{D}[c(\tau)]\; e^{-\frac{1}{\beta}S[q,a,b,c,\beta]} \quad , \\
S[q,a,b,c] &= S_2[q,a,b,c] + S_{int}[q,a,b,c] \quad , \\
S_2[q,a,b,c] &= \int_{-1}^{0}\mathrm{d}\tau\,\frac{1}{2}g_{ij}(x_f)\left(\dot{q}^i\dot{q}^j + a^ia^j + b^ic^j\right) \quad , \\
S_{int}[q,a,b,c] &= \int_{-1}^{0}\mathrm{d}\tau\,\left(\frac{1}{2}[g_{ij}(x(q)) - g_{ij}(x_f)](\dot{x}^i(q)\dot{x}^j(q) + a^ia^j + b^ic^j)\right. \\
&\quad \left. +\beta^2\left[V(x(q)) + V_{CT}(x(q))\right]\right) \quad .
\end{aligned}$$

So now we can calculate propagators from S_2 adding sources, performing Gaussian integrations, and making functional derivatives with respect to the sources as we did in Section 1.3.3; again we find that n-point propagators satisfy (Wick Theorem), and taking into account that we are in Euclidean and not real time and we have an additional $1/\beta$ factor that multiplies the action we find the following formulas[1]:

$$\begin{aligned}
\mathcal{K}(x_f,t_f;x_i,t_i) &= \langle\, 1\,\rangle e^{\frac{1}{2\beta}g_{ij}(x_f)\xi^i\xi^j}\langle\, e^{-\frac{1}{\beta}S_{int}}\,\rangle \quad , \\
\langle\, q^i(\tau)q^j(\sigma)\,\rangle &= -\beta g^{ij}(x_f)\Delta(\tau,\sigma) \quad , \\
\langle\, a^i(\tau)a^j(\sigma)\,\rangle &= \beta g^{ij}(x_f)\Delta_{gh}(\tau,\sigma) \quad , \\
\langle\, b^i(\tau)c^j(\sigma)\,\rangle &= -2\beta g^{ij}(x_f)\Delta_{gh}(\tau,\sigma) \quad .
\end{aligned}$$

Configuration Space Propagators in the continuum limit

We have to explain the meaning of $\Delta(\tau,\sigma)$ and $\Delta_{gh}(\tau,\sigma)$: as usual we read from S_2 that they are the inverse respectively of the kinetic operators

$$\begin{aligned}
K(\tau,\sigma) &= \delta(\tau-\sigma)\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} \quad , \\
K_{gh}(\tau,\sigma) &= \delta(\tau-\sigma) \quad ;
\end{aligned}$$

while the additional minus two factor in the $\langle\, bc\,\rangle$ propagator is due to Gaussian integration with anti-commuting variables. General solutions of equations

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\Delta(\tau,\sigma) &= \delta(\tau-\sigma) \quad , \\
\int_{-1}^{0}\delta(\tau-\tau')\Delta_{gh}(\tau',\sigma) &= \delta(\tau-\sigma)
\end{aligned}$$

in $\kappa([-1,0])^*$ are:

$$\begin{aligned}
\Delta(\tau,\sigma) &= \tau(\sigma+1)\theta(\tau-\sigma) + \sigma(\tau+1)\theta(\sigma-\tau) \quad , \\
\Delta_{gh}(\tau,\sigma) &= \delta(\tau-\sigma) \quad .
\end{aligned}$$

We immediately see that ghost propagator is proportional to a delta function, and thus it is divergent, as it is the propagator $\langle\, \dot{q}\dot{q}\,\rangle$, also proportional to $\delta(\tau-\sigma)$, as a matter of

[1] using the same notation as in Section 1.3.3.

fact ghost diagrams sum up to diagrams with $\dot{q}\dot{q}$ lines, and the resultant is a convergent diagram, as we will see. Using Fourier transform, that in $\kappa([-1,0])$ is a sine series, we can find the distributional meaning of $\theta(0)$ as an element of $\kappa([-1,0])^*$, repeating the procedure of Section 1.3.4, and we find $\theta(0) = 1/2$, so we can say that

$$\Delta(\tau,\tau) = \tau(\tau+1) \quad .$$

However even if we have found a distributional meaning of Δ and Δ_{gh}, we can not make explicit calculations yet, since we don't know how to manage products of distributions, and how to sum up divergent contributions to finite values (and the finite values they take): these calculations have to be made at the regulated level, and depend on the regularization chosen.

3.4: *Regularization schemes*

Now that we have developed the general structure of perturbative calculations in the continuum limit we can briefly review the results obtained in various regularization schemes. We are not going to develop algebraic calculations in the very detail, since a detailed analysis will be given in the next chapter for phase space path integrals. We perform calculations up to two loops, since at this order we fix univocally all the counterterms, as discussed in Section 3.3.2. In the controlling parameter β, a two-loop calculation is up to order β, so since in the exponential

$$e^{-\frac{1}{\beta}S_{int}}$$

we have a factor $1/\beta$, we need to expand S_{int} up to order β^2: explicitly we write $S_{int} = S_3 + S_4 + O(\beta^{5/2})$, where the index $_3$ or $_4$ means the square of the order in β of this part of the expansion. Now expanding the exponential

$$e^{-\frac{1}{\beta}S_{int}} = 1 - \frac{1}{\beta}S_{int} + \frac{1}{2\beta}S_{int}^2 + \dots \quad ,$$

and retaining only terms up to order β we obtain

$$e^{-\frac{1}{\beta}S_{int}} = 1 - \frac{1}{\beta}(S_3 + S_4) + \frac{1}{2\beta}S_3^2 + O(\beta^{3/2}) \quad ,$$

so performing the average and remembering that the average of a sum is the sum of averages we have

$$\langle\, e^{-\frac{1}{\beta}S_{int}} \,\rangle = 1 - \frac{1}{\beta}(\langle\, S_3 \,\rangle + \langle\, S_4 \,\rangle) + \frac{1}{2\beta}\langle\, S_3^2 \,\rangle + O(\beta^{3/2}) = e^{-\frac{1}{\beta}(\langle\, S_3 \,\rangle + \langle\, S_4 \,\rangle) + \frac{1}{2\beta}\langle\, S_3^2 \,\rangle_c} + O(\beta^{3/2})$$

where $\langle\, S_3^2 \,\rangle_c$ is the average of S_3^2 with only connected diagrams, *i.e.* without diagrams that are written as the product of two $\langle\, S_3 \,\rangle$ diagrams, in particular $\langle\, S_3^2 \,\rangle_c \equiv \langle\, S_3^2 \,\rangle - (\langle\, S_3 \,\rangle)^2$, since $(\langle\, S_3 \,\rangle)^2$ diagrams are counted in the second order expansion of the exponential.

Previous considerations, that are still valid for any regularization scheme, lead to an expression:

$$\mathcal{K}(x_f,t_f;x_i,t_i) = \langle\ 1\ \rangle e^{-\frac{1}{2\beta}g_{ij}(x_f)\xi^i\xi^j} e^{-\frac{1}{\beta}(\langle\ S_3\ \rangle+\langle\ S_4\ \rangle)+\frac{1}{2\beta}\langle\ S_3^2\ \rangle_c} + O(\beta^{3/2}) \quad ;$$

we know that the principal counterterm we have to fix is $V_{CT}(x)$, but since it is multiplied by β^2 in S_{int}, and each $x(\tau)$ is of order $\sqrt{\beta}$, as we will see in detail, we expand it to zeroth order around x_f, obtaining a function $V_{CT}(x_f)$, that has to be calculated imposing a renormalization condition on $\mathcal{K}$; furthermore, we already said that we have to fix also a global renormalization factor that has to take into account the proper definition of functional measures, as well as the rescaling of affine parameter t, and the determinants arising from Gaussian integration: this global factor is $A \equiv \langle\ 1\ \rangle$, and also it has to be fixed by the renormalization condition. A suitable renormalization condition is that $\psi(x_f,t_f)$, written as

$$\psi(x_f,t_f) = \int \mathrm{d}^D x_i \sqrt{g(x_i)}\mathcal{K}(x_f,t_f;x_i,t_i)\psi(x_i,t_i) \quad , \tag{3.4.1}$$

has to obey Schrödinger equation

$$-\frac{\mathrm{d}}{\mathrm{d}t_f}\psi(x_f,t_f) = \left(-\frac{1}{2}\nabla^2_{x_f} + V(x_f)\right)\psi(x_f,t_f) \quad .$$

Finally we can analyze the order in β of the variables q,a,b,c,ξ: from the propagators $\langle\ qq\ \rangle$, $\langle\ aa\ \rangle$ and $\langle\ bc\ \rangle$ we see immediately that any q,a,b or c counts as a factor $\sim \sqrt{\beta}$; and since in Eq. (3.4.1) we have to perform an integration in $\mathrm{d}^D x_i \equiv \mathrm{d}^D\xi$ and $\mathcal{K}$ provides a Gaussian factor

$$e^{-\frac{1}{2\beta}g_{ij}(x_f)\xi^i\xi^j}$$

we can construct a propagator $\langle\ \xi\xi\ \rangle$ again of order β, so also $\xi \sim \sqrt{\beta}$. So in fact we can expand $g(x)$, $V(x)$ and $V_{CT}(x)$ about x_f, obtaining for S_3 and S_4 the following results:

$$\begin{aligned}
S_3 &= \int_{-1}^{0}\mathrm{d}\tau\ \left[\frac{1}{2}g_{ij,k}(x_f)(q^k-\xi^k\tau)(\xi^i\xi^j - 2\xi^i\dot{q}^j + \dot{q}^i\dot{q}^j + a^ia^j + b^ic^j)\right] \quad , \\
S_4 &= \int_{-1}^{0}\mathrm{d}\tau\ \left[\frac{1}{4}g_{ij,kl}(x_f)(q^kq^l + \xi^k\xi^l\tau^2 - 2q^k\xi^l\tau)(\xi^i\xi^j - 2\xi^i\dot{q}^j + \dot{q}^i\dot{q}^j + a^ia^j + b^ic^j)\right. \\
&\quad \left. +\beta^2\big(V(x_f)+V_{CT}(x_f)\big)\right] \quad .
\end{aligned}$$

3.4.1: *Mode Regularization*

The basic idea of mode regularization[29, 30, 31, 32] is to rewrite every τ-dependent quantity in Fourier transform, that due to the boundaries is a sine expansion[1]:

$$\varphi^i(\tau) = \sum_{m=1}^{\infty}\varphi^i_m \sin(\pi m\tau) \quad ,$$

[1] since ghosts were introduced to exponentiate the determinant of the metric, we don't need them at endpoints, so we can impose also on them vanishing boundary conditions.

with φ is one of the fields q, a, b or c. We can say that integrating over all possible paths vanishing at the boundaries is equivalent to an integration over all values of Fourier coefficients, and as usual additional factors will be adsorbed in the global factor A. In that way we can explicitly integrate over τ, and obtain an action depending only on Fourier coefficients. Adding sources and performing Gaussian integration as usual determines propagators, that are written in the well-known form:

$$\begin{aligned}
\mathcal{K}(x_f, t_f; x_i, t_i) &= \langle\ 1\ \rangle e^{\frac{1}{2\beta} g_{ij}(x_f)\xi^i\xi^j} \langle\ e^{-\frac{1}{\beta}S_{int}}\ \rangle \quad , \\
\langle\ q^i(\tau)q^j(\sigma)\ \rangle &= -\beta g^{ij}(x_f)\Delta(\tau,\sigma) \quad , \\
\langle\ a^i(\tau)a^j(\sigma)\ \rangle &= \beta g^{ij}(x_f)\Delta_{gh}(\tau,\sigma) \quad , \\
\langle\ b^i(\tau)c^j(\sigma)\ \rangle &= -2\beta g^{ij}(x_f)\Delta_{gh}(\tau,\sigma) \quad ;
\end{aligned}$$

but now we have

$$\begin{aligned}
\Delta(\tau,\sigma) &= \sum_{m=1}^{\infty}\left[-\frac{2}{\pi^2 m^2}\sin(\pi m\tau)\sin(\pi m\sigma)\right] \quad , \\
\Delta_{gh}(\tau,\sigma) &= \sum_{m=1}^{\infty} 2\sin(\pi m\tau)\sin(\pi m\sigma) \quad .
\end{aligned}$$

An explicit calculation confirms that these are indeed the Fourier expansions of the distributions:

$$\begin{aligned}
\Delta(\tau,\sigma) &= \tau(\sigma+1)\theta(\tau-\sigma) + \sigma(\tau+1)\theta(\sigma-\tau) \quad , \\
\Delta_{gh}(\tau,\sigma) &= \delta(\tau-\sigma) \quad .
\end{aligned}$$

This far we haven't introduced a regularization yet, we only rewrote previous results in the space of Fourier modes. The regularization is effectively performed imposing a cutoff M in Fourier modes, and then taking the limit $M \to \infty$: explicitly we have

$$\varphi^i(\tau) = \lim_{M\to\infty}\sum_{m=1}^{M}\varphi^i_m \sin(\pi m\tau) \quad ,$$

and the regulated Δ and Δ_{gh} are now:

$$\begin{aligned}
\Delta(\tau,\sigma) &= \sum_{m=1}^{M}\left[-\frac{2}{\pi^2 m^2}\sin(\pi m\tau)\sin(\pi m\sigma)\right] \quad , \\
\Delta_{gh}(\tau,\sigma) &= \sum_{m=1}^{M} 2\sin(\pi m\tau)\sin(\pi m\sigma) \quad .
\end{aligned}$$

Let's see how to use such regulated propagators to perform perturbative calculations. As we did in Section 1.3.4 we write the derivative of a distribution $\Delta(\tau,\sigma)$ with respect to its first variable with $\overset{\bullet}{\Delta}$, a derivation with respect to the second variable with $\Delta\overset{\bullet}{}$. Using Wick theorem to perform the averages $\langle\ S_3\ \rangle$, $\langle\ S_4\ \rangle$ and $\langle\ S_3^2\ \rangle_c$ one finds expressions that involve integrals in $d\tau$, $d\sigma$ or $d\tau d\sigma$ of the Δ and Δ_{gh} distributions. Such integrals are ambiguous, or

divergent in the continuum limit, so have to be manipulated at the regulated level: even if we did not perform all the explicit calculations here we give an explicit example, the integral

$$I = \int_{-1}^{0} \mathrm{d}\tau \, \Delta|_\tau (\overset{\bullet}{\Delta}^{\bullet} + \Delta_{gh})|_\tau \quad ;$$

in the continuum limit it is divergent, since both $\overset{\bullet}{\Delta}^{\bullet}$ and Δ_{gh} are proportional to a delta function. However at the regulated level we immediately see that $\Delta_{gh} = {}^{\bullet\bullet}\Delta$, and that

$$\begin{aligned}
(\overset{\bullet}{\Delta}^{\bullet} + {}^{\bullet\bullet}\Delta)|_\tau &= \sum_{m=1}^{M} \left[-2\cos^2(\pi m \tau) + 2\sin^2(\pi m \tau) \right] \\
&= \partial_\tau \sum_{m=1}^{M} \left[-\frac{2}{\pi m} \sin(\pi m \tau)\cos(\pi m \tau) \right] \\
&= \partial_\tau \left(\overset{\bullet}{\Delta}|_\tau \right) \quad ;
\end{aligned}$$

so we can write

$$I = \lim_{M\to\infty} \int_{-1}^{0} \mathrm{d}\tau \, \Delta|_\tau \partial_\tau \left(\overset{\bullet}{\Delta}|_\tau \right) = \lim_{M\to\infty} \int_{-1}^{0} \mathrm{d}\tau \, \partial_\tau \left(\Delta|_\tau \right) \overset{\bullet}{\Delta}|_\tau \quad ,$$

and the integration by parts is justified since at regulated level both $\Delta|_\tau$ and $\overset{\bullet}{\Delta}|_\tau$ vanish at endpoint $\tau = -1, 0$. Finally the last integration can be computed directly in the continuum limit since it does not present any more ambiguities, and thus

$$I = \int_{-1}^{0} \mathrm{d}\tau \, \partial_\tau \left(\Delta|_\tau \right) \overset{\bullet}{\Delta}|_\tau = -2 \int_{-1}^{0} \mathrm{d}\tau \, \left(\tau + \frac{1}{2} \right)^2 = -\frac{1}{6} \quad ,$$

a perfectly unambiguous and finite result. However we emphasize the fact that this result depends on the tricks we used at regulated level, and therefore on the regularization scheme we use, in other schemes I could have a different result. Performing all calculation and solving ambiguities as we just explained, and finally imposing the renormalization condition one find the result

$$A = (2\pi\beta)^{-\frac{D}{2}} \, , \quad V_{MR}(x) = \frac{1}{8} R - \frac{1}{24} g^{ij} g^{mn} g_{kl} \Gamma^k{}_{im} \Gamma^l{}_{jn} \quad .$$

We report the explicit results of $\langle\, S_3 \,\rangle$, $\langle\, S_4 \,\rangle$ and $\langle\, S_3^2 \,\rangle_c$[26] since they will be useful to make

a confrontation with mode regularization results in phase space:

$$
\begin{aligned}
-\frac{1}{\beta}\langle\, S_3\,\rangle &= -\frac{1}{4\beta}\xi^k\xi^i\xi^j\partial_k g_{ij} \quad , \\
-\frac{1}{\beta}\langle\, S_4\,\rangle &= \frac{1}{24}\partial_k\partial_l g_{ij}\Big(\beta\big(g^{ij}g^{kl}-g^{ik}g^{jl}\big)-\xi^k\xi^l g^{ij}-\xi^i\xi^j g^{kl}+2\xi^i\xi^l g^{jk}-\frac{2}{\beta}\xi^i\xi^j\xi^k\xi^l\Big) \\
&\quad -\beta\Big(V+V_{MR}\Big) \quad , \\
\frac{1}{2\beta}\langle\, S_3^2\,\rangle_c &= \frac{1}{96}\partial_k g_{ij}\partial_l g_{mn}\Big(-\beta\Big(6g^{kl}g^{im}g^{jn}-4g^{km}g^{il}g^{jn}-g^{kl}g^{ij}g^{mn}+4g^{ik}g^{jl}g^{mn} \\
&\quad -4g^{ki}g^{lm}g^{jn}\Big)+4\xi^i\xi^m\Big(g^{jn}g^{kl}-g^{jl}g^{kn}\Big)+2\xi^k\xi^l g^{im}g^{jn} \\
&\quad +2\xi^i\xi^j\Big(2g^{km}g^{ln}-g^{kl}g^{mn}\Big)+4\xi^i\xi^k\Big(g^{jl}g^{mn}-2g^{jm}g^{ln}\Big) \\
&\quad +\frac{1}{\beta}\Big(\xi^i\xi^j\xi^m\xi^n g^{kl}-4\xi^i\xi^k\xi^m\xi^n g^{jl}+4\xi^i\xi^k\xi^l\xi^m g^{jn}\Big)\Big) \quad .
\end{aligned}
$$

3.4.2: *Dimensional regularization*

Dimensional regularization[33, 34, 35] is performed introducing S extra infinite regulating dimensions $\mathbf{t} = (t^1,\dots,t^S)$ that extends the "temporal" (affine) variable τ to $t^\mu(\tau,\mathbf{t})$ with $\mu = 0,1,\dots,S$ and $\mathrm{d}^{S+1}t = \mathrm{d}\tau\mathrm{d}^S\mathbf{t}$, so the action in $S+1$ dimensions reads

$$
S = \int_\Omega \mathrm{d}^{S+1}t\,\left[\frac{1}{2}g_{ij}(\partial_\mu x^i\partial_\mu x^j + a^i a^j + b^i c^j) + \beta^2(V+V_{DR})\right] \quad ,
$$

where $\Omega = [-1,0]\times\mathbb{R}^S$. Using VBC variables qs instead of xs and performing the usual split $S = S_2 + S_{int}$ we obtain

$$
\begin{aligned}
S_2 &= \int_\Omega \mathrm{d}^{S+1}t\,\frac{1}{2}g_{ij}(x_f)(\partial_\mu q^i\partial_\mu q^j + a^i a^j + b^i c^j) \quad , \\
S_{int} &= \int_\Omega \mathrm{d}^{S+1}t\,\left[\frac{1}{2}(g_{ij}(x)-g_{ij}(x_f))(\partial_\mu x^i\partial_\mu x^j + a^i a^j + b^i c^j) + \beta^2(V+V_{DR})\right] \quad .
\end{aligned}
$$

In this case the Fourier transform will be a series for $t^0 \equiv \tau$, but an integral for $\mathbf{t}$, so at a regulated level the propagators would read

$$
\begin{aligned}
\langle\, q^i(t)q^j(s)\,\rangle &= -\beta g^{ij}(x_f)\Delta(t,s) \quad , \\
\langle\, a^i(t)a^j(s)\,\rangle &= \beta g^{ij}(x_f)\Delta_{gh}(t,s) \quad , \\
\langle\, b^i(t)c^j(s)\,\rangle &= -2\beta g^{ij}(x_f)\Delta_{gh}(t,s) \quad ;
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta(t,s) &= \int\frac{\mathrm{d}^S\mathbf{k}}{(2\pi)^S}\sum_{m=1}^{\infty}\frac{-2}{\pi^2 m^2}\sin(\pi m\tau)\sin(\pi m\sigma)\,e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} \quad , \\
\Delta_{gh}(t,s) &= \int\frac{\mathrm{d}^S\mathbf{k}}{(2\pi)^S}\sum_{m=1}^{\infty}2\sin(\pi m\tau)\sin(\pi m\sigma)\,e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} = \delta(\tau-\sigma)\delta^S(\mathbf{t}-\mathbf{s}) \equiv \delta^{S+1}(t,s).
\end{aligned}
$$

In the limit $S \to 0$ these propagators reduce to the usual

$$\Delta(\tau,\sigma) = \tau(\sigma+1)\theta(\tau-\sigma) + \sigma(\tau+1)\theta(\sigma-\tau) \quad ,$$
$$\Delta_{gh}(\tau,\sigma) = \delta(\tau-\sigma) \quad .$$

However we note that at the regulated level holds the following equality:

$$\partial_\mu^2 \Delta(t,s) = \Delta_{gh}(t,s) = \delta^{S+1}(t,s) \quad ;$$

this allows us to solve ambiguities and divergencies using manipulations at the regulated level and then taking the limit $S \to 0$, as we see calculating the following integral[1]:

$$\begin{aligned}
J &= \int_{-1}^{0} \mathrm{d}\tau \int_{-1}^{0} \mathrm{d}\sigma \, \dot{\Delta}\Delta^{\bullet}\dot{\Delta}^{\bullet} \\
&\to \int \mathrm{d}^{S+1}t \int \mathrm{d}^{S+1}s \, {}_{\mu}\Delta\Delta_{\nu} \, {}_{\mu}\Delta_{\nu} \\
&= \frac{1}{2}\int \mathrm{d}^{S+1}t \int \mathrm{d}^{S+1}s \, {}_{\mu}\Delta \quad {}_{\mu}(\Delta_{\nu})^2 \\
&= -\frac{1}{2}\int \mathrm{d}^{S+1}t \int \mathrm{d}^{S+1}s \, {}_{\mu\mu}\Delta(\Delta_{\nu})^2 \\
&= -\frac{1}{2}\int \mathrm{d}^{S+1}t \int \mathrm{d}^{S+1}s \, \delta^{S+1}(t,s) \, (\Delta_{\nu})^2 \\
&= -\frac{1}{2}\int \mathrm{d}^{S+1}t \, (\Delta_{\nu})^2\Big|_t \\
&\to -\frac{1}{2}\int_{-1}^{0} \mathrm{d}\tau \, (\dot{\Delta}^{\bullet})^2\Big|_{\tau} = -\frac{1}{24} \quad ;
\end{aligned}$$

while in mode regularization this integral would result $-1/12$, so indeed fixing ambiguous results is a procedure regularization-dependent.

Performing all perturbative calculations up to order β one finds that dimensional regularization counterterm is generally covariant; explicitly

$$A = (2\pi\beta)^{-\frac{D}{2}} \, , \quad V_{DR}(x) = \frac{1}{8}R \quad .$$

3.4.3: *Time slicing*

In the time-slicing scheme we don't need renormalization conditions, since is the time slicing procedure that led to a path integral formulation in the first time: in particular if we recover expression (3.2.2), integrate out momenta with a Gaussian integration for each k, add ghosts to exponentiate the metric determinant; then split discretized action S_N into $S_{2,N} + S_{int,N}$, add sources and perform perturbative calculations up to order β always remaining at the discretized level, everything is unambiguous and we don't need any counterterm, in fact it is sufficient the Weyl ordering term

$$V_{TS}(x) = \frac{1}{8}(R + g^{ij}\Gamma^{k}{}_{il}\Gamma^{l}{}_{jk}) \quad .$$

[1] we introduce the notation ${}_{\mu}\Delta$ and Δ_{μ} to denote derivation with respect to the first or second $S+1$ dimensional variable of Δ.

The other difference between time slicing and other regularization schemes is the presence of a global multiplying factor $g(x_i)^{-1/4}g(x_f)^{1/4}$,explicitly

$$\mathcal{K}(x_f,t_f;x_i,t_i) = (2\pi\beta)^{-\frac{D}{2}}\frac{g(x_f)^{1/4}}{g(x_i)^{1/4}}e^{-\frac{1}{2\beta}g_{ij}\xi^i\xi^j}\left\langle\, e^{-S_{int}}\,\right\rangle \quad ,$$

where the average is performed for a fixed ϵ and then the limit $\epsilon \to 0$ is taken. So with the terminology used so far we can say that time-slicing counterterm is the Weyl ordering potential V_{TS}, while its global renormalization factor is $A = (2\pi\beta)^{-D/2}g(x_i)^{-1/4}g(x_f)^{1/4}$; however we emphasize that in time slicing we don't need to impose renormalization conditions, or solve any ambiguity, everything is well defined and derives from the very definition of $\mathcal{K}$ as a transition element and the application of the midpoint rule with Weyl ordered operators.

If we perform phase space calculations in time slicing, *i.e.* we don't integrate out momenta in Eq. (3.2.2), but we directly add sources for ps and qs (no ghosts are needed), we can evaluate again the transition element up to order β with discretized phase space propagators: explicit calculations[26] show that even if we have different diagrams, they can be seen as combinations of configuration space diagrams; as a matter of fact nothing changes, and again the only "counterterms" are V_{TS} and $A = (2\pi\beta)^{-D/2}g(x_i)^{-1/4}g(x_f)^{1/4}$.

Chapter 4

Regularizations in phase space

"Minucioso, inmóvil, secreto, urdió
en el tiempo su alto laberinto
invisible."

Jorge Luis Borges

As we have seen in the previous chapter, we can make a path integral formulation of the transition element $\langle x_f, t_f | x_i, t_i \rangle$ if the Hamiltonian function that describes the affine parameter evolution of the system through Schrödinger equation is the quantized relativistic Hamiltonian

$$\mathcal{H}(\hat{x}, \hat{p}) = \frac{1}{2} g(\hat{x})^{-\frac{1}{4}} \hat{p}_i g(\hat{x})^{\frac{1}{2}} g^{ij}(\hat{x}) \hat{p}_j g(\hat{x})^{-\frac{1}{4}} + V(\hat{x}) \quad .$$

We showed that this function derives from the Gupta-Bleuler quantization of a classical action describing a particle of mass $m \geq 0$ moving along a geodesic[1] in a curved space with metric $g_{ij}(x)$. In this chapter we will briefly discuss how such a model is very useful even to calculate one-loop effective actions for scalar fields in a gravitational background.

Then we focus on the main feature of this work: the treatment of the integral kernel $\mathcal{K}$ as a phase space path integral. We will show that in such a case every perturbative calculation is well defined even in the continuum limit, so any regularization scheme leads to the same results. In order to do that we perform explicit calculations in the space of Fourier coefficients (introducing a cutoff M in the Fourier modes in the limit $M \to \infty$ we have mode regularization). However we will see that the presence of the "tadpole" $ip\dot{x}$ in the free action complicates the structure of propagators and Wick theorem.

[1] with an Euclidean affine parametrization.

4.1: *Worldline methods on curved space*

Before we deal with phase space path integrals we make a little digression, and discuss worldline methods we introduced in Section 2.2 in the presence of gravity.

We saw in Sections 2.2.1 and 2.2.2 that if we have a field coupled to an external background field we could write the one-loop effective action as a quantum mechanical path integral. Consider now a real scalar field in a D-dimensional space-time ϕ coupled to the metric $g_{\mu\nu}$. The Euclidean quantum field action reads

$$S[\phi, g] = \int \mathrm{d}^D x \sqrt{g} \frac{1}{2} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 + \xi R \phi^2) \quad ,$$

where m is the mass of the scalar particle and ξ is a possible non-minimal coupling to the scalar curvature R. The Euclidean one-loop effective action $\Gamma[g]$ describes all possible one-loop graphs with the scalar field in the loop and any number of gravitons on the external legs. However the fluctuation of the metric, *i.e.* the graviton, is not quantized. As we saw in Section 2.2.1 we have

One-loop Effective Action for a scalar field in a gravitational background by means of Path Integral

$$\Gamma[g] = -\ln \mathrm{Det}^{-1/2} (-\nabla^2 + m^2 + \xi R) \quad ,$$

where ∇^2 is the covariant Laplacian; now repeating the same procedure that led to Eq. (2.2.3) we obtain

$$\Gamma[g] = -\frac{1}{2} \int_0^\infty \frac{\mathrm{d}T}{T} \int_{PBC} \mathscr{D}[x(t)]\, e^{-S[x,g]} \quad ,$$

where the action $S[x, g]$ is

$$S[x, g] = \int_0^T \mathrm{d}t \left(\frac{1}{4} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + m^2 + \xi R(x) \right) \quad ;$$

so we see this is exactly the action we have for our relativistic Hamiltonian if we set $V(x) = m^2 + \xi R(x)$. So the path integral of Eq. (3.3.1)[1] is useful to calculate the effective action of a scalar field in a gravitational background by means of a fictitious quantum mechanics.

However we would have obtained the same result for $\Gamma[g]$ first quantizing the scalar particle model not yet gauge-fixed we described in Section 3.1.1: as a matter of fact the action (3.1.5) in Euclidean time and adding the non-minimal coupling to scalar curvature reads

$$S[x, e] = \int \mathrm{d}t \frac{1}{2} \left[e^{-1} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + e(m^2 + \xi R(x)) \right] \quad ;$$

we quantized this theory in a Gupta-Bleuler way, alternatively we could have quantized it directly by means of a path integral over the fields x and e, taking care of the volume of the

[1] in this case we wrote the configuration space path integral, however we could as well write the phase space path integral.

gauge group by a Fadeev-Popov gauge fixing procedure, obtaining the same result for $\Gamma[g]$, namely

$$\Gamma[g] = \int_{PBC} \frac{\mathscr{D}[x]\mathscr{D}[e]}{\text{Vol(Gauge)}} e^{-S[x,e]} \quad ,$$

where the division with the volume of the gauge group denotes formally the Fadeev-Popov gauge fixing procedure[14].

We conclude that the "fictitious" quantum mechanics is not so fictitious: it corresponds to the first quantization of the relativistic scalar particle which makes the loop in the effective action.

4.2: *Phase Space*

In section 3.4.3 we stated that time slicing calculations in phase space led to a fixing of A and V_{ph} at a regulated level. However we want to show that in fact a regularization is not necessary, because every calculation could be as well performed in the continuum limit. Let's start from the relativistic Hamiltonian of our model

$$\mathcal{H}(p,x) = \frac{1}{2} g^{ij} p_i p_j + V(x) \quad ; \tag{4.2.1}$$

where x and p are the n-dimensional coordinate and conjugate momentum vectors, and $V(x)$ is a scalar function of the coordinates only. We know that the Kernel is given by a phase space path integral in Euclidean time:

$$\mathcal{K}(x_f, t_f; x_i, t_i) \equiv \int_{\substack{x(t_i)=x_i \\ x(t_f)=x_f}} \mathscr{D}[x(t)]\mathscr{D}[p(t)] e^{-\widetilde{S}_E(p(t),x(t))} \quad , \tag{4.2.2}$$

where $\widetilde{S}_E(p,x)$ is the Euclidean time action.

Since the theory described by such an Hamiltonian as (4.2.1) is super-renormalizable, we have only to fix a global renormalization constant, we will call it as usual A, and a coordinate function $V_{ph}(x)$, correcting the potential $V(x)$, to univocally determine the Kernel at any perturbative order. Since these two terms are fixed at the first order in $\beta = t_f - t_i$, we will evaluate the Kernel at this order[1] and fix A and $V_{ph}(x)$.

As we did in the previous chapter we explicit the dependence on the difference $t_f - t_i \equiv \beta$: in order to do so we set $t = t_f + \beta\tau$, and define $\widehat{S}_E \equiv \beta \widetilde{S}_E$, obtaining[2]:

$$\mathcal{K}(x_f, t_f; x_i, t_i) \equiv \int_{\substack{x(t_i)=x_i \\ x(t_f)=x_f}} \mathscr{D}[x(t(\tau))]\mathscr{D}[p(t(\tau))] e^{-\frac{1}{\beta}\widehat{S}_E(p(t),x(t))} \quad ,$$

$$\widehat{S}_E = \int_{-1}^{0} \mathrm{d}\tau \left(\frac{1}{2}\beta^2 g^{ij} p_i p_j - i\beta p\dot{x} + \beta^2 \Big(V(x) + V_{ph}(x) \Big) \right) \quad .$$

[1] this is known as "one-loop calculation of the transition amplitude".

[2] we omit the time dependence of x and p in $\widehat{S}_E$ to lighten the notation.

We redefine $p \to \frac{1}{\beta} p$ [1] and set $x^i = x_f^i - \underbrace{(x_i^i - x_f^i)}_{\xi^i} \tau + q^i(\tau)$ [2] , finally obtaining[3]:

$$\mathcal{K}(x_f, t_f; x_i, t_i) \equiv \int\limits_{VBC} \mathscr{D}[q(\tau)]\mathscr{D}[p(\tau)] e^{-\frac{1}{\beta} S(p(\tau), q(\tau))} \quad , \tag{4.2.3}$$

$$S = \int_{-1}^{0} \mathrm{d}\tau \left(\frac{1}{2} g^{ij} p_i p_j - i\beta p_i \dot{q}^i + i p_i \xi^i + \beta^2 \Big(V(x) + V_{ph}(x) \Big) \right) \quad . \tag{4.2.4}$$

4.2.1: *Mode expansion*

In order to find propagators and use perturbative methods, we split as usual the action S in an interacting S_{int} and a free S_2 part. So $S = S_2 + S_{int}$, with:

$$S_2 = \int_{-1}^{0} \mathrm{d}\tau \left(\frac{1}{2} g^{ij}(x_f) p_i p_j - i\beta p_i \dot{q}^i + i p_i \xi^i \right) \quad , \tag{4.2.5}$$

$$S_{int} = \int_{-1}^{0} \mathrm{d}\tau \left(\frac{1}{2} \Big(g^{ij}(x) - g^{ij}(x_f) \Big) p_i p_j + \beta^2 \Big(V(x) + V_{ph}(x) \Big) \right) \quad ; \tag{4.2.6}$$

where $g^{ij}(x_f)$ is the metric calculated at final point, and $g^{ij}(x)$, $V(x)$ and $V_{ph}(x)$ are intended as functions of the $q(\tau)$ path. So defining usual averages over the free action

$$\langle\, \mathscr{F}[p(\tau), x(\tau)] \,\rangle \equiv \frac{\int\limits_{VBC} \mathscr{D}[q(\tau)]\mathscr{D}[p(\tau)]\, \mathscr{F}[p(\tau), x(\tau)]\, e^{-\frac{1}{\beta} S_2(p(\tau), q(\tau))}}{\int\limits_{VBC} \mathscr{D}[q(\tau)]\mathscr{D}[p(\tau)] e^{-\frac{1}{\beta} S_2(p(\tau), q(\tau))}} \quad , \tag{4.2.7}$$

apart from

$$\langle\, 1 \,\rangle \equiv \int\limits_{VBC} \mathscr{D}[q(\tau)]\mathscr{D}[p(\tau)] e^{-\frac{1}{\beta} S_2(p(\tau), q(\tau))} \quad , \tag{4.2.8}$$

where $\mathscr{F}[p(\tau), q(\tau)]$ is an arbitrary functional of coordinate and momentum paths; we can write $\mathcal{K}(x_f, t_f; x_i, t_i)$ as:

$$\mathcal{K}(x_f, t_f; x_i, t_i) = \langle\, 1 \,\rangle \langle\, \exp\{-\beta^{-1} S_{int}\} \,\rangle \quad . \tag{4.2.9}$$

Since the exponential can be written as a power series, all we have to do is find an expression for n-point propagators $\langle\, \phi^{i_1}(\tau_1) \cdots \phi^{i_n}(\tau_n) \,\rangle$ where each ϕ can be either a p or a q, since is straightforward to see that the average of a sum equals the sum of the averages. Furthermore we will see that in that case the 1-point propagator doesn't vanish, nevertheless every

[1] this redefinition can be adsorbed in the definition of momenta measure $\mathscr{D}[p(t)]$, and affects the path integral only for an overall multiplying constant, *i.e.* it will be adsorbed in the definition of the renormalization constant A.

[2] the transformation from x to q respects the boundary conditions if $q(-1) = q(0) = 0$, *i.e.* with VBC. Such a transformation does not affect $\mathcal{D}[x(t(\tau))] \equiv \mathcal{D}[q(\tau)]$ because path measures are translationally invariant.

[3] from now on we will omit that the τ-dependence of paths pass through $t(\tau)$, and simply write $x(\tau)$ and $p(\tau)$

n-point propagator can be written as a function of 1 and 2-point propagators only, in a modified version of Wick Theorem; effectively all we have to find are 1 and 2-point propagators. Before we delve into the calculation of propagators we find very useful to expand coordinate and momentum paths in Fourier transform, in order to properly define the measure $\mathscr{D}[q(\tau)]\mathscr{D}[p(\tau)]$ and perform the integration over $\mathrm{d}\tau$.

We define the covariant measure as:

$$\mathscr{D}[q(\tau)]\mathscr{D}[p(\tau)] \equiv \prod_{-1<\tau<0} \prod_i \mathrm{d}p_i(\tau)\mathrm{d}q^i(\tau) \quad .$$

Now, since $q(\tau)$ vanish at the boundaries, its Fourier expansion will be written in terms of sines, and since Hamilton equations tell us that momenta are proportional to the derivative of coordinates, we will expand $p(\tau)$ with cosines[1]:

$$p_i(\tau) = \sum_{m=0}^{\infty} p_{im} \cos(\pi m \tau) \quad , \tag{4.2.10}$$

$$q^i(\tau) = \sum_{m=1}^{\infty} q^i_m \frac{\sin(\pi m \tau)}{\pi m} \quad . \tag{4.2.11}$$

So the path integral measure reduces to integration over all values of the Fourier coefficients[2]:

$$\mathscr{D}[q(\tau)]\mathscr{D}[p(\tau)] \equiv \prod_{-1<\tau<0} \prod_i \mathrm{d}p_i(\tau)\mathrm{d}q^i(\tau) = \prod_{i=1}^{N} \prod_{m=1}^{\infty} \mathrm{d}p_{i0}\mathrm{d}p_{im}\mathrm{d}q^i_m \quad .$$

We are therefore able to perform integration over τ on the free action S_2, substituting (4.2.10) and (4.2.11) in $p(\tau)$ and $q(\tau)$. We have to use the following integrals:

$$\int_{-1}^{0} \mathrm{d}\tau \, \cos(\pi m \tau) = 0 \quad ,$$
$$\int_{-1}^{0} \mathrm{d}\tau \, = 1 \quad ,$$
$$\int_{-1}^{0} \mathrm{d}\tau \, \cos(\pi m \tau) \cos(\pi n \tau) = \frac{\delta_{mn}}{2} \quad ;$$

and obtain:

$$S_2 = \frac{1}{4} g^{ij}(x_f) \sum_{m=1}^{\infty} p_{im} p_{jm} - \frac{i}{2} \sum_{m=1}^{\infty} p_{im} q^i_m + \left(\frac{1}{2} g^{ij}(x_f) p_{i0} p_{j0} + i p_{i0} \xi^i \right) \quad . \tag{4.2.12}$$

[1] momentum expansion presents one more term than coordinate one, the *zero mode term*, which vanish for the qs due to their sines expansion (*i.e.* due to their vanishing boundary conditions). This additional term requires to be treated apart from the others, and we will see it is responsible for the non-vanishing of the $\langle\, p \,\rangle$ and $\langle\, pp \,\rangle$ propagators. The idea of expanding momenta with cosines was introduced in [36], in the case of flat space-time.

[2] this procedure bring a normalization factor, that we will adsorb in the definition of A, so we will omit it here.

As already stated in the footnotes, we will treat the momentum *zero mode* apart from the rest of the free action, so we split $S_2 = S_2^n + S_2^0$ with:

$$S_2^n \equiv \frac{1}{4} g^{ij}(x_f) \sum_{m=1}^{\infty} p_{im} p_{jm} - \frac{i}{2} \sum_{m=1}^{\infty} p_{im} q_m^i \quad ,$$
$$S_2^0 \equiv \frac{1}{2} g^{ij}(x_f) p_{i0} p_{j0} + i p_{i0} \xi^i \quad .$$

4.2.2: *Propagators*

We are now ready to calculate the one and two-point propagators from the free action. The easiest way is to define a *generating functional* $Z[\Phi]$:

$$Z[\Phi] = \int \mathcal{D}[\phi] e^{-\frac{1}{\beta}(S(\phi) - \phi \cdot \Phi)} \quad ,$$

for any ϕ-dependent functional action, where ϕ is a covariant or contravariant vector function and Φ is respectively a contravariant or covariant vector function. $Z[\Phi]$ is very useful to calculate propagators, in fact we know very well that the n-point propagator, for an action $S(\phi)$, can be calculated as:

$$\langle\, \phi^{i_1} \cdots \phi^{i_n} \,\rangle = \left(\frac{1}{Z[\Phi]} \beta^n \frac{\delta}{\delta \Phi_{i_n}} \cdots \frac{\delta}{\delta \Phi_{i_1}} Z[\Phi] \right) \Bigg|_{\Phi=0} \quad ,$$
$$\langle\, 1 \,\rangle = Z[\Phi] \Big|_{\Phi=0} \quad ,$$

where $\frac{\delta}{\delta \Phi}$ is a functional derivative. Since in our case we already performed the time integral in the action, the action reduces to a function of the momentum and coordinate Fourier coefficients, and functional derivatives reduce to ordinary derivatives. So we define[1] $S_2^s \equiv \sum_m \left(P_m^i p_{im} + Q_{im} q_m^i \right) + P_0^i p_{i0}$, and $Z[P,Q] = \lim_{M \to \infty} Z_m[P,Q]$ where

$$\begin{aligned} Z_m[P,Q] &\equiv \int \prod_{i=1}^{N} \prod_{m=1}^{M} \mathrm{d}p_{i0} \mathrm{d}p_{im} \mathrm{d}q_m^i \; e^{-\frac{1}{\beta}(S_2 - S_2^s)} \\ &= \left(\int \prod_i \mathrm{d}p_{i0} \; e^{-\frac{1}{\beta}(S_2^0 - S_2^{s0})} \right) \left(\int \prod_i \prod_m \mathrm{d}p_{im} \mathrm{d}q_m^i \; e^{-\frac{1}{\beta}(S_2^n - S_2^{sn})} \right) \\ &\equiv Z_0[P_0] Z_m[P_m, Q_m] \quad . \end{aligned} \tag{4.2.13}$$

The calculation of $Z_m[P,Q]$ is quite easy. We define a $2nm$-dimensional vector

$$\zeta^\mu \equiv (\, q_1^1 \,,\, q_1^2 \,, \ldots \,,\, q_1^n \,,\, p_{11} \ldots \,,\, p_{n1} \,,\, q_2^1 \,, \ldots \,,\, p_{n2} \,, \ldots \,,\, p_{nm} \,) \quad ,$$

so the measure $\prod_i \prod_m \mathrm{d}p_{im} \mathrm{d}q_m^i \equiv \prod_\mu \zeta^\mu$, and $Z_m[P_m, Q_m] \equiv Z_m[Z_\mu]$, where:

$$Z_m[Z_\mu] = \int \prod_\mu \mathrm{d}\zeta^\mu \, e^{-\frac{1}{\beta} S(\zeta, Z)} \quad , \tag{4.2.14}$$

[1] P and Q are called *sources*.

with

$$S(\zeta, Z) = \frac{1}{2} K_{\mu\nu}\zeta^\mu\zeta^\nu - Z_\mu\zeta^\mu \quad , \tag{4.2.15}$$

$$K_{\mu\nu} \equiv \begin{pmatrix} \mathbf{H} & 0 & \ldots\ldots & \\ 0 & \mathbf{H} & 0 & \ldots \\ 0 & \ldots & \ddots & 0 \\ 0 & \ldots\ldots & & \mathbf{H} \end{pmatrix} \quad 2nm \times 2nm \text{ matrix} , \tag{4.2.16}$$

$$\mathbf{H} \equiv \begin{pmatrix} 0 & -\frac{i}{2}\mathbf{1} \\ -\frac{i}{2}\mathbf{1} & \frac{1}{2}\mathbf{g}^{-1}(x_f) \end{pmatrix} \quad 2n \times 2n \text{ matrix} . \tag{4.2.17}$$

This is a well known Gaussian integral[1], and the result depends only on $K \equiv \det K_{\mu\nu}$ and $K^{\mu\nu} \equiv \left(K^{-1}\right)^{\mu\nu}$, so since K will enter in our definition of the renormalization factor A and furthermore cancels out in an average like (4.2.7), we need only to know $K^{\mu\nu}$ to calculate propagators. An elementary calculation yields:

$$K^{\mu\nu} = \begin{pmatrix} \mathbf{H}^{-1} & 0 & \ldots\ldots\ldots & \\ 0 & \mathbf{H}^{-1} & 0 & \ldots \\ 0 & \ldots & \ddots & 0 \\ 0 & \ldots\ldots\ldots & & \mathbf{H}^{-1} \end{pmatrix} \quad , \tag{4.2.18}$$

$$\mathbf{H}^{-1} = \begin{pmatrix} 2i\mathbf{g}^{-1}(x_f) & 2i\mathbf{1} \\ 2i\mathbf{1} & 0 \end{pmatrix} \quad . \tag{4.2.19}$$

So we are now able to perform Gaussian integration and calculate $Z_m[Z_\mu]$:

$$Z_m[Z_\mu] = \int \prod_\mu \mathrm{d}\zeta^\mu \, \exp\left\{-\frac{1}{\beta}\left(\frac{1}{2}K_{\mu\nu}\zeta^\mu\zeta^\nu - Z_\mu\zeta^\mu\right)\right\} = \left(\beta^{nm}K^{-\frac{1}{2}}\right) e^{\frac{1}{2\beta}K^{\mu\nu}Z_\mu Z_\nu} ; \tag{4.2.20}$$

restoring proper P_m and Q_m dependencies we obtain:

$$Z_m[P_m, Q_m] = \left(\beta^{nm}K^{-\frac{1}{2}}\right) \exp\left\{\frac{1}{\beta}\sum_{m=1}^{M}\left(g^{ij}(x_f)Q_{im}Q_{jm} + 2iQ_{im}P^i_m\right)\right\} \quad . \tag{4.2.21}$$

We remain to consider $Z_0[P_0]$, but that time we have to proceed more carefully. In that case we have:

$$S_2^0 - S_2^{0s} = \frac{1}{2}g^{ij}(x_f)p_{i0}p_{j0} + ip_{i0}\xi^i - P_0^k p_{k0} \quad ;$$

so we are dealing with a Gaussian Integral somewhat similar to (4.2.20), but with an "effective" source term $(P_0^i - i\xi^i)p_{i0}$, that brings a term $\frac{1}{2\beta}g_{ij}(x_f)(P_0^i - i\xi^i)(P_0^j - i\xi^j)$ in the

[1] see Appendix C of Part I for a discussion of such integrals.

exponential after integration[1]. So we have performed every integration, and the resulting generating functional is:

$$Z_m[P,Q] = \widetilde{A}\exp\left\{\frac{1}{2\beta}\left[\sum_{m=1}^{M}\left(2g^{ij}(x_f)Q_{im}Q_{jm} + 4iQ_{im}P_m^i\right) + g_{ij}\left(P_0^i - i\xi^i\right)\left(P_0^j - i\xi^j\right)\right]\right\} \quad ;$$

where $\widetilde{A} = \beta^{nm+1}K^{-\frac{1}{2}}g(x_f)^{\frac{1}{2}}$. Now with that convention is straightforward to see that $\langle\ 1\ \rangle = \widetilde{A}\exp\left\{-\frac{1}{2\beta}g_{ij}(x_f)\xi^i\xi^j\right\}$, but in defining path integral measure we omitted some normalization constants, promising to adsorb them later in the definition of the global renormalization constant A. It is now time to remember that promise and define A in a way that[2]:

$$\langle\ 1\ \rangle \equiv Ag^{\frac{1}{4}}(x_f)g^{-\frac{1}{4}}(x_i)\,e^{-\frac{1}{2\beta}g_{ij}(x_f)\xi^i\xi^j} \quad , \tag{4.2.22}$$

so the generating functional has to be modified accordingly and becomes:

$$\begin{aligned} Z_m[P,Q] = A\frac{g(x_f)^{\frac{1}{4}}}{g(x_i)^{\frac{1}{4}}}\exp\left\{\frac{1}{2\beta}\left[\sum_{m=1}^{M}\left(2g^{ij}(x_f)Q_{im}Q_{jm} + 4iQ_{im}P_m^i\right) + \right.\right. \\ \left.\left. + g_{ij}(x_f)\left(P_0^i - i\xi^i\right)\left(P_0^j - i\xi^j\right)\right]\right\} \quad . \end{aligned} \tag{4.2.23}$$

We are therefore ready to make derivatives with respect to the sources and calculate propagators. Since we are dealing with exponentials of quadratic forms of sources (*i.e.* each term is a product of two sources), except for the zero-mode term, which present a linear term ($-ig_{ij}P_0^i\xi^j$), it is easy to prove that odd-point propagators vanish for p_ms and q_ms, and $(n > 2)$-point propagators are obtained from the 2-point one performing all possible *Wick contractions* in a standard way; however odd-point p_0 propagators don't vanish, but it is easy to prove by induction that every propagator depends only on the 1 and 2-point ones in the following manner[3]:

The Wick Theorem has to be modified in Phase Space Path Integral

$$\begin{aligned} \langle\ pp\ \rangle &= \langle\ p\ \rangle\langle\ p\ \rangle + \langle\ pp\ \rangle_c \quad , \\ \langle\ ppp\ \rangle &= \langle\ p\ \rangle\langle\ p\ \rangle\langle\ p\ \rangle + 3\langle\ p\ \rangle\langle\ pp\ \rangle_c \quad , \\ \langle\ pppp\ \rangle &= \langle\ p\ \rangle\langle\ p\ \rangle\langle\ p\ \rangle\langle\ p\ \rangle + 6\langle\ p\ \rangle\langle\ p\ \rangle\langle\ pp\ \rangle_c + 3\langle\ pp\ \rangle_c\langle\ pp\ \rangle_c \quad , \\ \langle\ ppppp\ \rangle &= \langle\ p\ \rangle\langle\ p\ \rangle\langle\ p\ \rangle\langle\ p\ \rangle\langle\ p\ \rangle + \cdots \quad , \\ &\cdots \quad . \end{aligned}$$

[1] since we immediately know the inverse of $g^{ij}(x_f)$ to be $g_{ij}(x_f)$.

[2] the reason to extract the factor $g^{\frac{1}{4}}(x_f)g^{-\frac{1}{4}}(x_i)$ defining A is a matter of convenience, as we will see in a few pages.

[3] note that this rule applies even for the p_m and q_m propagators, noting that their 1-point propagator vanishes.

An easy calculation then provides:

$$\langle\, p_{i0} \,\rangle = -ig_{ij}(x_f)\xi^j \quad , \tag{4.2.24}$$
$$\langle\, q^i_m q^j_n \,\rangle = 2\beta g^{ij}(x_f)\delta_{mn} \quad , \tag{4.2.25}$$
$$\langle\, p_{im} q^j_n \,\rangle = 2\beta \delta_i{}^j \delta_{mn} \quad , \tag{4.2.26}$$
$$\langle\, p_{im} p_{jn} \,\rangle = 0 \quad , \tag{4.2.27}$$
$$\langle\, p_{i0} p_{j0} \,\rangle_c = \beta g_{ij}(x_f) \quad ; \tag{4.2.28}$$

all other two point propagators vanish. Finally, remembering (4.2.10) and (4.2.11) we obtain

$$\langle\, p_i(\tau) \,\rangle = -ig_{ij}\xi^j \quad , \tag{4.2.29}$$
$$\langle\, q^i(\tau) q^j(\sigma) \,\rangle = -\beta g^{ij}\Delta(\tau,\sigma) \quad , \tag{4.2.30}$$
$$\langle\, p_i(\tau) q^j(\sigma) \,\rangle = -i\beta \delta_i{}^j \,{}^\bullet\Delta(\tau,\sigma) \quad , \tag{4.2.31}$$
$$\langle\, p_i(\tau) p_j(\sigma) \,\rangle_c = \beta g_{ij} \quad ; \tag{4.2.32}$$

where[1]

$$\Delta(\tau,\sigma) \equiv \sum_{m=1}^{\infty} -\frac{2}{\pi^2 m^2} \sin(\pi m\tau)\sin(\pi m\sigma) \quad , \tag{4.2.33}$$
$${}^\bullet\Delta(\tau,\sigma) \equiv \sum_{m=1}^{\infty} -\frac{2}{\pi m} \cos(\pi m\tau)\sin(\pi m\sigma) \quad . \tag{4.2.34}$$

Phase Space Propagators

As we saw in the previous chapter, an explicit calculation yields that they are the Fourier series expression respectively of the distributions

$$\Delta(\tau,\sigma) = \tau(\sigma+1)\theta(\tau-\sigma) + \sigma(\tau+1)\theta(\sigma-\tau) \quad ,$$
$${}^\bullet\Delta(\tau,\sigma) = \sigma + \theta(\tau-\sigma) \quad .$$

Since in $\kappa([-1,0])^*$ we have that $\theta(0) = 1/2$, we then have the following expressions for distributions at coincident times:

$$\Delta(\tau,\tau) = \tau(\tau+1) \quad ,$$
$${}^\bullet\Delta(\tau,\tau) = \tau + \frac{1}{2} \quad .$$

If we introduce a cutoff M in the Fourier modes we obtain the mode regulated propagators in phase space:

$$\Delta_M(\tau,\sigma) \equiv \sum_{m=1}^{M} -\frac{2}{\pi^2 m^2} \sin(\pi m\tau)\sin(\pi m\sigma) \quad ,$$
$${}^\bullet\Delta_M(\tau,\sigma) \equiv \sum_{m=1}^{M} -\frac{2}{\pi m} \cos(\pi m\tau)\sin(\pi m\sigma) \quad ;$$

in the limit $M \to \infty$ they become the two distributions Δ and ${}^\bullet\Delta$.

[1] the symbol ${}^\bullet\Lambda(x,y)$ is intended to be the derivative with respect to the first parameter (x in this case) of Λ, $\Lambda^\bullet$ the derivative with respect to the second parameter. Henceforth we will always use this notation to mean the derivative of Δ with respect to τ or σ (so the propagator $\langle\, q^i(\tau) p_j(\sigma) \,\rangle = -i\beta\delta^i_j {}^\bullet\Delta$).

4.2.3: *Perturbative expansion*

All we have to do now, to fix A and $V_{ph}(x)$, is to calculate, up to order β, $\langle\, \exp\{-\beta^{-1}S_{int}\}\,\rangle$. In order to do that we expand $g^{ij}(x)$ about x_f, and obtain $S_{int} = S_3 + S_4 + \ldots$ [1] , with:

$$S_3 = \int_{-1}^{o} \mathrm{d}\tau \left[\frac{1}{2}\partial_k g^{ij}\left(q^k p_i p_j - \tau \xi^k p_i p_j\right)\right] \quad , \tag{4.2.35}$$

$$S_4 = \int_{-1}^{0} \mathrm{d}\tau \left[\frac{1}{4}\partial_l\partial_k g^{ij}\left(q^l q^k p_i p_j - 2\tau\xi^l q^k p_i p_j + \tau^2\xi^l\xi^k p_i p_j\right) + \beta^2\left(V + V_{ph}\right)\right] \quad ; \tag{4.2.36}$$

where g^{ij} , V and V_{ph} are calculated at final point x_f. Since $\exp\{-\beta^{-1}S_{int}\} = 1 - \beta^{-1}S_{int} + \frac{1}{2}\beta^{-2}S_{int}^2 + \ldots$ and $S_3 + S_4 = O(\beta^{\frac{3}{2}}) + O(\beta^2)$, to order β we need to calculate:

$$\begin{aligned}\langle\, \exp\{-\beta^{-1}S_{int}\}\,\rangle &= 1 - \beta^{-1}\langle\, S_3\,\rangle - \beta^{-1}\langle\, S_4\,\rangle + \frac{1}{2}\beta^{-2}\langle\, S_3^2\,\rangle + O(\beta^2) \\ &\equiv \exp\{-\beta^{-1}(\langle\, S_3\,\rangle + \langle\, S_4\,\rangle) + \frac{1}{2}\beta^{-2}\langle\, S_3^2\,\rangle_c\} + O(\beta^2) \quad ,\end{aligned}$$

where $\langle\, S_3^2\,\rangle_c$ is the average containing only connected diagrams, *i.e.* $\langle\, S_3^2\,\rangle_c \equiv \langle\, S_3^2\,\rangle - \langle\, S_3\,\rangle^2$. We write down explicitly S_3^2 :

$$\begin{aligned}S_3^2 = \int_{-1}^{0}\int_{-1}^{0} \mathrm{d}\tau\,\mathrm{d}\sigma \Big[&\frac{1}{4}\partial_k g^{ij}\partial_l g^{mn}\left(q^k q^l p_i p_j p_m p_n - \tau\xi^k p^l p_i p_j p_m p_n\right. \\ &\left.-\sigma q^k\xi^l p_i p_j p_m p_n + \tau\sigma\xi^k\xi^l p_i p_j p_m p_n\right)\Big] \quad .\end{aligned} \tag{4.2.37}$$

Now we can calculate $\langle\, S_3\,\rangle$, $\langle\, S_4\,\rangle$ and $\langle\, S_3\,\rangle_c$. After applying contraction rules for n-point propagators described in the previous section, we need only to perform some integrations over τ and σ involving Δ functions. As a matter of fact such integrals are perfectly well-defined in the continuum limit, since we don't have nor divergent terms, nor ambiguous products of distributions; all the integrals are here reported[2]:

$$\mathbf{I_1} = \int_{-1}^{0} \mathrm{d}\tau\, \dot{\Delta}\big|_\tau = 0$$

$$\mathbf{I_2} = \int_{-1}^{0} \mathrm{d}\tau\, \tau = -\frac{1}{2}$$

$$\mathbf{I_3} = \int_{-1}^{0} \mathrm{d}\tau\, \dot{\Delta}\big|_\tau = 0$$

[1] Subscripts $_3$ and $_4$ in the expansion stands for the square of the order in β of these parts of the action. In fact note that each p , q and ξ weigh as $\sqrt{\beta}$.

[2] see Appendix E for examples of explicit calculations.

$$\mathbf{I_4} = \int_{-1}^{0} \mathrm{d}\tau\, \Delta\big|_{\tau} = -\frac{1}{6}$$
$$\mathbf{I_5} = \int_{-1}^{0} \mathrm{d}\tau\, \dot{\Delta}\big|_{\tau}^{2} = \frac{1}{12}$$
$$\mathbf{I_6} = \int_{-1}^{0} \mathrm{d}\tau\, \tau\dot{\Delta}\big|_{\tau} = \frac{1}{12}$$
$$\mathbf{I_7} = \int_{-1}^{0} \mathrm{d}\tau\, \tau^2 = \frac{1}{3}$$
$$\mathbf{I_8} = \int_{-1}^{0} \mathrm{d}\tau \int_{-1}^{0} \mathrm{d}\sigma\, \Delta = -\frac{1}{12}$$
$$\mathbf{I_9} = \int_{-1}^{0} \mathrm{d}\tau \int_{-1}^{0} \mathrm{d}\sigma\, \dot{\Delta}\big|_{\tau}\, \dot{\Delta} = \frac{1}{12}$$
$$\mathbf{I_{10}} = \int_{-1}^{0} \mathrm{d}\tau \int_{-1}^{0} \mathrm{d}\sigma\, \dot{\Delta}\, \Delta^{\bullet} = -\frac{1}{12}$$
$$\mathbf{I_{11}} = \int_{-1}^{0} \mathrm{d}\tau \int_{-1}^{0} \mathrm{d}\sigma\, \dot{\Delta}\big|_{\tau}\, \Delta^{\bullet}\big|_{\sigma} = 0$$
$$\mathbf{I_{12}} = \int_{-1}^{0} \mathrm{d}\tau \int_{-1}^{0} \mathrm{d}\sigma\, \tau\dot{\Delta} = \frac{1}{12}$$
$$\mathbf{I_{13}} = \int_{-1}^{0} \mathrm{d}\tau \int_{-1}^{0} \mathrm{d}\sigma\, \tau\, \Delta^{\bullet}\big|_{\sigma} = 0$$
$$\mathbf{I_{14}} = \int_{-1}^{0} \mathrm{d}\tau \int_{-1}^{0} \mathrm{d}\sigma\, \tau\sigma = \frac{1}{4}$$

where $\big|_{\tau,\sigma}$ means that the Δ function is calculated in $\tau = \sigma$ and so reduces to a single-variable function, and every non-simmetric integral with respect to τ and σ has its inverted counterpart which brings the same result. We write explicitly $\langle\, S_3 \,\rangle$ and $\langle\, S_4 \,\rangle$ in dependence of such integrals, for the sake of completeness[1]:

$$\begin{aligned}
\langle\, S_3 \,\rangle &= \int_{-1}^{0} \mathrm{d}\tau\, \frac{1}{2}\partial_k g^{ij}\Big(\langle\, q^k p_i \,\rangle\langle\, p_j \,\rangle + \langle\, q^k p_j \,\rangle\langle\, p_i \,\rangle - \tau\xi^k \langle\, p_i p_j \,\rangle\Big) \\
&= -\frac{1}{2}\partial_k g^{ij}\Big(2\beta\delta^k{}_i g_{jl}\xi^l \mathbf{I_1} - \xi^k \mathbf{I_2}\big(\beta g_{ij} - \xi^l\xi^m g_{il}g_{jm}\big)\Big) \quad , \\
\langle\, S_4 \,\rangle &= \int_{-1}^{0} \mathrm{d}\tau \Big[\frac{1}{4}\partial_l\partial_k g^{ij}\Big(\langle\, q^l q^k \,\rangle\langle\, p_i p_j \,\rangle + \langle\, q^l p_i \,\rangle\langle\, q^k p_j \,\rangle + \langle\, q^l p_j \,\rangle\langle\, q^k p_i \,\rangle \\
&\quad - 2\tau\xi^l\big(\langle\, q^k p_i \,\rangle\langle\, p_j \,\rangle + \langle\, q^k p_j \,\rangle\langle\, p_i \,\rangle\big) + \tau^2\xi^l\xi^k\langle\, p_i p_j \,\rangle\Big) + \beta^2\Big(V + V_{ph}\Big)\Big] \\
&= -\frac{1}{4}\partial_l\partial_k g^{ij}\Big(\beta g^{lk}\mathbf{I_3}\big(\beta g_{ij} - \xi^m\xi^n g_{im}g_{jn}\big) + 2\beta^2\delta^l{}_i\delta^k{}_j\mathbf{I_4} - 4\xi^l\xi^m\delta^k{}_i g_{jm}\mathbf{I_5} \\
&\quad - \xi^l\xi^k\mathbf{I_6}\big(\beta g_{ij} - \xi^m\xi^n g_{im}g_{jn}\big)\Big) + \beta^2\Big(V + V_{ph}\Big) \quad .
\end{aligned}$$

[1] $\langle\, S_3^2 \,\rangle_c$ takes an analogous form, but since there a lot of different contractions with 4-point propagators, we omit it here.

Finally, after some boring calculations[1], we obtain:

$$-\frac{1}{\beta}\langle\ S_3\ \rangle = -\frac{1}{4\beta}\xi^k\xi^i\xi^j\partial_k g_{ij} + \frac{1}{2}\xi^j\Gamma^i{}_{ij} \quad , \tag{4.2.38}$$

$$\begin{aligned}-\frac{1}{\beta}\langle\ S_4\ \rangle = {}& \frac{1}{24}\partial_k\partial_l g_{ij}\Big(\beta(g^{ij}g^{kl} - g^{ik}g^{jl}) - \xi^k\xi^l g^{ij} - \xi^i\xi^j g^{kl} + 2\xi^k\xi^j g^{il} - \frac{2}{\beta}\xi^i\xi^j\xi^k\xi^l\Big) \\ & - \beta\Big(V + V_{ph}\Big) + \frac{1}{8}\xi^k\xi^l g^{ij}\partial_k\partial_l g_{ij} \quad ,\end{aligned} \tag{4.2.39}$$

$$\begin{aligned}\frac{1}{2\beta}\langle\ S_3^2\ \rangle_c = {}& \frac{1}{96}\partial_k g_{ij}\partial_l g_{mn}\Big(\beta\Big(g^{ij}g^{kl}g^{mn} - 2g^{ij}g^{kn}g^{lm} - 2g^{ik}g^{jl}g^{mn} + 4g^{ik}g^{jm}g^{ln} \\ & + 8g^{il}g^{jm}g^{kn} - 4g^{im}g^{jn}g^{kl}\Big) + 4\xi^i\xi^m\Big(g^{jn}g^{kl} - g^{jl}g^{kn}\Big) + 2\xi^k\xi^l g^{im}g^{jn} \\ & + 2\xi^i\xi^j\Big(2g^{km}g^{ln} - g^{kl}g^{mn}\Big) + 4\xi^i\xi^k\Big(g^{jl}g^{mn} - 2g^{jm}g^{ln}\Big) \\ & + \frac{1}{\beta}\Big(\xi^i\xi^j\xi^m\xi^n g^{kl} - 4\xi^i\xi^k\xi^m\xi^n g^{jl} + 4\xi^i\xi^k\xi^l\xi^m g^{jn}\Big)\Big) \\ & - \frac{1}{8}\xi^k\xi^l g^{im}g^{jn}\partial_k g_{ij}\partial_l g_{mn} \quad ;\end{aligned} \tag{4.2.40}$$

where $\Gamma^i{}_{ij}$ is the connection coefficient calculated at x_f. Now it will be very useful to isolate the cyan terms we have explicited from the rest of the average, so from now on when we write $-\frac{1}{\beta}\langle\ S_3\ \rangle$, $-\frac{1}{\beta}\langle\ S_4\ \rangle$ and $\frac{1}{2\beta}\langle\ S_3^2\ \rangle_c$ we will consider the ones above *without* cyan terms; the latter will be written apart as $\exp\left\{\frac{1}{2}\xi^j\Gamma^i{}_{ij} + \frac{1}{4}\xi^i\xi^j\partial_i\Gamma^k{}_{kj}\right\}$ since from the definition of connection coefficients $\partial_i\Gamma^k{}_{kj} \equiv \frac{1}{2}\Big(g^{ij}\partial_k\partial_l g_{ij} - g^{im}g^{jn}\partial_k g_{ij}\partial_l g_{mn}\Big)$.

4.2.4: *Fixing A and V_{ph}*

We have calculated all the averages we needed, so now we are able to fix A and V_{ph} once and for all. In order to do that we compare our $\mathcal{K}_{ph}(x_f,t_f;x_i,t_i)$[2] at order β with the configuration space, mode regulated one that we already figured out previously. Since the kernel does not depend on the regularization chosen, nor whether we perform our calculations in configuaration or phase space, it has to be satisfied the following condition:

$$\mathcal{K}^{MR}_{ph}(x_f,t_f;x_i,t_i) \equiv \mathcal{K}^{MR}_{conf}(x_f,t_f;x_i,t_i) \quad .$$

[1] remembering that since

$$\partial_k\Big(g^{ia}g_{aj}\Big) = \partial_k\partial_l\Big(g^{ia}g_{aj}\Big) = 0 \quad ,$$

then we have:

$$\begin{aligned}\partial_k g^{ij} &= -g^{im}g^{jn}\partial_k g_{mn} \quad , \\ \partial_k\partial_l g^{ij} &= -g^{im}g^{jn}\partial_k\partial_l g_{mn} + g^{im}g^{jq}g^{np}\partial_l g_{mn}\partial_k g_{pq} + g^{im}g^{jq}g^{np}\partial_k g_{mn}\partial_l g_{pq} \quad .\end{aligned}$$

Furthermore all $-\beta^{-1}\langle\ S_4\ \rangle$ terms derived from last two terms of this last expression are included in $(2\beta)^{-1}\langle\ S_3^2\ \rangle_c$, and modify accordingly the result to the one reported.

[2] $_{ph}$ is added to distinguish between differently calculated kernels.

Now our phase space kernel up to order β can be written as:

$$\mathcal{K}^{MR}_{ph}(x_f,t_f;x_i,t_i) = Ae^{-\frac{1}{2\beta}g_{ij}\xi^i\xi^j}\frac{g^{\frac{1}{4}}(x_f)}{g^{\frac{1}{4}}(x_i)}e^{\frac{1}{2}\xi^j\Gamma^i{}_{ij}+\frac{1}{4}\xi^i\xi^j\partial_i\Gamma^k{}_{kj}}e^{-\frac{1}{\beta}\left(\langle\, S_3\,\rangle+\langle\, S_4\,\rangle\right)+\frac{1}{2\beta}\langle\, S_3^2\,\rangle_c}+O(\beta^2) \tag{4.2.41}$$

where metric and connection coefficients are calculated at the final point if not otherwise specified. The last thing to point out to make the comparison straightforward is that the following relation is satisfied:

$$g^{\frac{1}{4}}(x_i) = g^{\frac{1}{4}}(x_f)e^{\frac{1}{2}\xi^j\Gamma^i{}_{ij}+\frac{1}{4}\xi^i\xi^j\partial_i\Gamma^k{}_{kj}}+O(\beta^2) \quad .$$

Proof. Expanding $g^{\frac{1}{4}}(x_i)$ around x_f up to second order[1] in $\xi \equiv x_i - x_f$ we obtain:

$$\begin{aligned}
g^{\frac{1}{4}}(x_i) &= g^{\frac{1}{4}}(x_f)+\xi^i\partial_i g^{\frac{1}{4}}(x_f)+\frac{1}{2}\xi^i\xi^j\partial_i\partial_j g^{\frac{1}{4}}(x_f)+O(\beta^2)\\
&= g^{\frac{1}{4}}(x_f)\left(1+\xi^i\frac{\partial_i g^{\frac{1}{4}}(x_f)}{g^{\frac{1}{4}}(x_f)}+\frac{1}{2}\xi^i\xi^j\frac{\partial_i\partial_j g^{\frac{1}{4}}(x_f)}{g^{\frac{1}{4}}(x_f)}\right)+O(\beta^2)\\
&= g^{\frac{1}{4}}(x_f)\left(1+\xi^i\frac{\partial_i g^{\frac{1}{4}}(x_f)}{g^{\frac{1}{4}}(x_f)}+\frac{1}{2}\xi^i\xi^j\partial_i\frac{\partial_j g^{\frac{1}{4}}(x_f)}{g^{\frac{1}{4}}(x_f)}+\frac{1}{8}\xi^i\xi^j\frac{\partial_i g(x_f)}{g(x_f)}\frac{\partial_j g^{\frac{1}{4}}(x_f)}{g^{\frac{1}{4}}(x_f)}\right)+O(\beta^2)\\
&= g^{\frac{1}{4}}(x_f)\Bigg(1+\xi^i\partial_i\ln g^{\frac{1}{4}}(x_f)+\frac{1}{2}\xi^i\xi^j\partial_i\Big(\partial_j\ln g^{\frac{1}{4}}(x_f)\Big)\\
&\qquad\qquad+\frac{1}{8}\xi^i\xi^j\partial_i\Big(\ln g^{\frac{1}{4}}(x_f)\Big)\partial_j\Big(\ln g^{\frac{1}{4}}(x_f)\Big)\Bigg)+O(\beta^2)
\end{aligned}$$

but since $\partial_i \ln\det\mathbf{A} = \mathrm{tr}\left(\mathbf{A}^{-1}\partial_i\mathbf{A}\right)$,

$$\begin{aligned}
&= g^{\frac{1}{4}}(x_f)\left(1+\frac{1}{4}\xi^i g^{jk}\partial_i g_{jk}+\frac{1}{8}\xi^i\xi^j\partial_i\left(g^{jk}\partial_j g_{jk}\right)+\frac{1}{32}\xi^i\xi^j g^{jk}\partial_i g_{jk}\,g^{lm}\partial_j g_{lm}\right)+O(\beta^2)\\
&= g^{\frac{1}{4}}(x_f)\left(1+\frac{1}{2}\xi^i\Gamma^j{}_{ji}+\frac{1}{4}\xi^i\xi^j\partial_i\Gamma^k{}_{kj}+\frac{1}{8}\xi^i\xi^j\Gamma^k{}_{ki}\Gamma^l{}_{lj}\right)+O(\beta^2)
\end{aligned}$$

now changing dummy index accordingly, and noting that $\exp\left\{\frac{1}{2}\xi^j\Gamma^i{}_{ij}+\frac{1}{4}\xi^i\xi^j\partial_i\Gamma^k{}_{kj}\right\} \equiv 1+\frac{1}{2}\xi^i\Gamma^j{}_{jk}+\frac{1}{4}\xi^i\xi^j\partial_i\Gamma^k{}_{kj}+\frac{1}{8}\xi^i\xi^j\Gamma^k{}_{ki}\Gamma^l{}_{lj}+O(\beta^2)$, we obtain:

$$g^{\frac{1}{4}}(x_i) = g^{\frac{1}{4}}(x_f)e^{\frac{1}{2}\xi^j\Gamma^i{}_{ij}+\frac{1}{4}\xi^i\xi^j\partial_i\Gamma^k{}_{kj}}+O(\beta^2) \quad .$$

□

So Eq.(4.2.41) reduces to:

$$\mathcal{K}^{MR}_{ph}(x_f,t_f;x_i,t_i) = Ae^{-\frac{1}{2\beta}g_{ij}\xi^i\xi^j}e^{-\frac{1}{\beta}\left(\langle\, S_3\,\rangle+\langle\, S_4\,\rangle\right)+\frac{1}{2\beta}\langle\, S_3^2\,\rangle_c}+O(\beta^2) \quad ; \tag{4.2.42}$$

[1] since each ξ counts as $\sqrt{\beta}$ in a power expansion.

so a straightforward comparison with $\mathcal{K}^{MR}_{conf}(x_f, t_f; x_i, t_i)$ fixes A and V_{ph} as:

$$A = (2\pi\beta)^{-\frac{n}{2}} \quad , \tag{4.2.43}$$

$$V_{ph}(x) = -\frac{R(x)}{8} + \frac{1}{8} g^{ij}(x)\Gamma^k{}_{il}(x)\Gamma^l{}_{jk}(x) \quad . \tag{4.2.44}$$

We note that mode-regulated, phase space counterterm is identical to the same one in time-slicing, as we expected.

4.2.5: *Regularization is not needed for perturbative calculations*

We have just proved that introducing a regularization scheme is not very useful if one calculates perturbative corrections in phase space. In fact we were able to solve all integrals without introducing a cutoff M in the Fourier modes, *i.e.* without recurring to mode regularization. Furthermore we obtained the same result as performing calculations in time slicing. As a matter of fact phase space path integral counterterm V_{ph} is just the term due to the Weyl ordering of quantum Hamiltonian, necessary to evaluate transition elements at intermediate times; furthermore A can be fixed univocally imposing a suitable consistency condition: we imposed the equality of $\mathcal{K}_{ph}$ with the one calculated in configuration space mode regularization at order β, since it is the most general consistency condition we could require.

We note a peculiar fact: while in time slicing counterterms in phase and configuration space are exactly the same, this is not true in mode regularization; we already pointed out that integrating out the momenta in the continuum limit was a dangerous and not well defined manipulation: it leads to ambiguities and divergencies, to the introduction of ghosts and finally to a modification of the counterterms even if we treat the same regularization scheme. Thus one could try to derive the mode regularization configuration space path integral from the phase space one integrating out the momenta at the regulated level, and try to understand where the modifications to the counterterm arise.

Appendices

Appendix D

Weyl Ordering

We want to prove the following

Theorem D.0.1. *If we define M as*

$$M \equiv \int \mathrm{d}^D p \, \langle z|p|O_W(x,p)\rangle\langle p|y\rangle \quad ,$$

where $O_W(x,p)$ is an operator in Weyl ordered form, then

$$M \equiv \int \mathrm{d}^D p \, \langle z|p\rangle O_W\left(\frac{1}{2}(z+y),p\right)\langle p|y\rangle \quad .$$

where now $O_W\left(\frac{1}{2}(z+y),p\right)$ is the function corresponding to $O_W(x,p)$ when the operator x is substituted with $(z+y)/2$ and the operator p with the variable p.

Proof. An operator O_W in Weyl ordering can be written in Taylor series of x and p, thus it will contain in general the symmetrized expression of terms $x^m p^r$ with arbitrary m and r. If we call such symmetrized expressions $(x^m p^r)_S$, the following formula holds:

$$(x^m p^r)_S = \frac{1}{2^m} \sum l = 0m \binom{m}{l} x^{m-l} p^r x^l \quad ;$$

so the transition element $M = \langle z|(x^m p^r)_S|y\rangle$ can be written as:

$$\begin{aligned} M &= \int \mathrm{d}^D p \, \frac{1}{2^m} \sum l = 0m \binom{m}{l} z^{m-l} y^l p^r \langle z|p\rangle\langle p|y\rangle \\ &= \int \mathrm{d}^D p \left(\frac{z+y}{2}\right)^m p^r \langle z|p\rangle\langle p|y\rangle \quad ; \end{aligned}$$

since this is true for any m and r the theorem is proved. □

As it concerns the Trotter-like approximation

$$\int \mathrm{d}^D p_k \, e^{ip_k\cdot(x_k-x_{k-1})} \left(e^{-\epsilon\mathcal{H}(\bar{x}_{k-1/2},p_k)}\right)_W = \int \mathrm{d}^D p_k \, e^{ip_k\cdot(x_k-x_{k-1})} \, e^{-\epsilon\mathcal{H}_W(\bar{x}_{k-1/2},p_k)} + O(\epsilon^{\frac{3}{2}}) \quad ,$$

it is justified since the difference between $(\exp(-\epsilon\mathcal{H}))_W$ and $\exp(-\epsilon\mathcal{H}_W)$ consists of terms without a p, surely of higher order in ϵ, and terms with at least one p, however if we split the Hamiltonian in a free and interacting part and calculate phase space propagators $\langle\, p_{k,i}p_{k,j}\,\rangle$ and $\langle\, p_{k,i}\bar{x}^j_{l-1/2}\,\rangle$ they are both of order unity, the formula holds at first order in ϵ.

Appendix E

Distributional integration in phase space

The distributional integrals I_1–I_{14} that we have to perform in Section 4.2.3 are not ambiguous and quite easy to solve if we remember the continuum limit expression of Δ and ${}^{\bullet}\Delta$

As an example we explicitly calculate

$$I_{10} = \int_{-1}^{0} \mathrm{d}\tau \int_{-1}^{0} \mathrm{d}\sigma \, {}^{\bullet}\Delta\Delta^{\bullet} \quad ,$$

that since in the continuum limit we have

$$\begin{aligned} {}^{\bullet}\Delta(\tau,\sigma) &= \sigma + \theta(\tau-\sigma) \quad , \\ \Delta^{\bullet}(\tau,\sigma) &= \tau + \theta(\sigma-\tau) \quad , \end{aligned}$$

becomes

$$\begin{aligned} I_{10} &= \int_{-1}^{0} \mathrm{d}\tau \int_{-1}^{0} \mathrm{d}\sigma \, (\sigma + \theta(\tau-\sigma))(\tau + \theta(\sigma-\tau)) \\ &= \iint \tau\sigma + 2\int \tau(\tau+1) + \iint \theta(\tau-\sigma)\theta(\sigma-\tau) \quad , \end{aligned}$$

where $\int\int$ stands for integration from -1 to 0 both in τ and σ, $\int$ integration from -1 to 0 only in τ. The second factor of the last expression comes from the terms $\int\int(\sigma\theta(\sigma-\tau))$ and $\int\int(\tau\theta(\tau-\sigma))$ that are symmetric and thus give the same result[1]. Now the only integral that deserve some thinking is $\int\int \theta(\tau-\sigma)\theta(\sigma-\tau)$, however since the function $\theta(\tau-\sigma)\theta(\sigma-\tau) \neq 0$ in a set of null measure the integral vanishes[2]. A straightforward calculation the

[1] clearly the integration $\int \mathrm{d}\sigma\theta(\tau-\sigma)$ gives the factor $\tau+1$ that multiplies τ

[2] problems could be present at the boundaries due to oscillating factors with infinite frequency, but surely at the regulated level we don't have such a problem. In fact even in the limit $M \to \infty$ an explicit calculation using the Fourier series of $\theta(t-s)$ we gave in Section 1.3.4 (in this case $\beta = 1$) yields zero.

leads to:

$$I_{10} = \int_{-1}^{0} d\tau \int_{-1}^{0} d\sigma \, \dot{\Delta}\dot{\Delta} = -\frac{1}{12} \quad .$$

The other integrals are even simpler than this one.

Conclusions

We have proved that the phase space path integral describing the affine evolution dictated by a relativistic Hamiltonian, *i.e.* the quantum motion of a particle in curved space, is independent of the regularization scheme used to define the functional measure, in contrast with the configuration space path integral that is explicitly dependent on the regularization scheme chosen, as far as we perform perturbative calculations.

In particular, if we treat this path integral as a one dimensional QFT, we have a super-renormalizable theory that needs the introduction of ghosts, in configuration space, in order to manage divergencies; and a regularization scheme as well as a suitable counterterm in order to solve ambiguities and obtain the same result in any case for the integral kernel of affine evolution at any order. Since in the continuum limit we can obtain the configuration space functional integral from the phase space one formally integrating out the momenta, we can say that such formal manipulation is quite problematic: it leads to divergent and ambiguous diagrams and thus to the necessity to introduce ghosts and counterterms depending on the regularization scheme.

Phase space path integral instead is well defined in the continuum limit, *i.e.* it gives the same perturbative results for any regularization scheme chosen to define precisely the functional measure of momentum and coordinate paths: in fact apart a global factor that we could adsorb in the definition of such functional measure, the only modification we need to introduce in the relativistic action is an additional factor that arises from the particular ordering (the Weyl ordering) we need to give to the quantum Hamiltonian operator to evaluate its transition element between a position and a momentum eigenstate.

The only subtlety in the phase space path integral formulation is that since there is a non-vanishing momentum zero-mode linear term in the free action, the structure of Wick theorem is different from the usual one, since odd-points correlation functions don't vanish; however it is possible to see that any correlation function can be written by means of the one and two-point propagators.

The integration of momenta at a regulated level in time slicing does not modify the structure of the counterterm: in fact in both configuration and phase space path integral time-slicing has the same counterterm. Instead mode regularization counterterm in configuration space is different from the one in phase space; it would be worth to analyze the way such modifications to phase space counterterm arise when we integrate out the momenta at the regulated level: in order to derive unambiguously mode regulated configuration path integral from the phase space one.

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"Di questa poesia
mi resta
quel nulla
d'inesauribile segreto"