Exercise sheet 1

Nonlinear Dispersive PDEs
Sommersemester 2019
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Exercise 1 (5pt). Properties of the Fourier transform

Prove that for any $f, g \in \mathscr{S}(\mathbb{R}^d)$, and any $\alpha \in \mathbb{N}^d$:

- $\left(\frac{i\partial}{2\pi}\right)^\alpha \hat{f} = \widehat{\left(x^\alpha f\right)}$;
- $(ix)^\alpha \hat{f} = \widehat{\left(\left(\frac{\partial}{2\pi}\right)^\alpha f\right)}$;
- $\widehat{(f * g)} = \hat{f} \cdot \hat{g}$;
- $\widehat{(f \cdot g)} = \hat{f} * \hat{g}$.

Exercise 2 (10pt). Properties of $|\cdot|^\lambda$

Prove that for any $\lambda \in \mathbb{R}, f(x) = |x|^\lambda$ does not belong to $L^1(\mathbb{R}^d)$, and discuss whether the function fails to be integrable in a neighbourhood of zero or infinity, depending on the value of λ.

Exercise 3 (4pt). Gaussian Mollifier

Prove that $\left(\frac{1}{\pi\varepsilon}\right)^{\frac{d}{2}} e^{-\frac{1}{\varepsilon}|\cdot|^2}$ is a mollifier, *i.e.* that for any measurable function f such that $\exists N \in \mathbb{N}$, $(1 + |\cdot|)^{-N} f(\cdot) \in L^1(\mathbb{R}^d)$:

$$\left(\frac{1}{\pi\varepsilon}\right)^{\frac{d}{2}} \left(e^{-\frac{1}{\varepsilon}|\cdot|^2} * f\right)(x) \underset{\varepsilon\to 0}{\longrightarrow} f(x) \qquad \text{(for almost every } x \in \mathbb{R}^d) .$$

Exercise 4 (6pt). Compactly supported distributions

Prove that any $T \in \mathscr{D}'(\mathbb{R}^d)$ with $\operatorname{supp}(T)$ compact is tempered (*i.e.* $T \in \mathscr{S}'(\mathbb{R}^d)$).

Exercise 5 (5pt). Density argument

Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators in $L^2(\mathbb{R})$, such that:

- for all $n \in \mathbb{N}$, $\|U_n\|_{\mathscr{L}(L^2,L^2)} = 1$[1];
- $\|(U - U_n)\varphi\|_2 \underset{n\to\infty}{\to} 0$ for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$, with $\|U\|_{\mathscr{L}(L^2,L^2)} = 1$.

Prove that U_n converges strongly to U, *i.e.* that for all $f \in L^2(\mathbb{R}^d)$,

$$\lim_{n\to\infty} \|(U - U_n)f\|_2 = 0 .$$

[*Hint:* Use the density of $\mathscr{S}$ in L^2 to approximate f with a sequence $(\varphi_m)_{m\in\mathbb{N}} \subset \mathscr{S}$, and then use an $\frac{\varepsilon}{2}$-argument.]

[1]The operator norm $\|T\|_{\mathscr{L}(L^2,L^2)} := \sup_{\|f\|_2=1} \|Tf\|_2$. It follows that for all $f \in L^2(\mathbb{R}^d)$, $\|Tf\|_2 \leq \|T\|_{\mathscr{L}(L^2,L^2)} \|f\|_2$.