Exercise 1 (5pt). HLS Inequality

Prove the Hardy-Littlewood-Sobolev inequality (Corollary i.5 of the lecture):

$$\forall 0 < \alpha < d, \forall (p, r) \in ]1, \infty[^2 \text{ such that } \frac{1}{p} + \frac{\alpha}{d} = 1 + \frac{1}{r} :$$

$$\exists C > 0, \forall f \in L^p(\mathbb{R}^d), \| \cdot |^{-\alpha} * f \|_r \leq C \| f \|_p .$$

You may use the refined Young’s inequality.

Exercise 2 (4pt). Weakly integrable but not integrable functions

Prove that for any $1 \leq p < \infty$, $L^p(\mathbb{R}^d) \subset L^p_w(\mathbb{R}^d)$ is a strict inclusion. (Hint: think of $| \cdot |^2$)

Exercise 3 (6pt). The weak $L^p$ norm is a quasi-norm

Verify that the weak $L^p$ norm, defined as

$$\| g \|_{p,w} := \sup_{\lambda > 0} \lambda^p \int_{\{|g(x)| > \lambda\}} dx$$

satisfies the properties of a quasi-norm (the triangle inequality is true up to a constant).

Exercise 4 (7pt). Properties of the Fourier transform

Prove that for any $f, g \in \mathcal{S}(\mathbb{R}^d)$, and any $\alpha \in \mathbb{N}^d$:

- $\left( \frac{i \partial}{2\pi} \right)^\alpha \hat{f} = (x^\alpha f)$;
- $(ix)^\alpha \hat{f} = \left( \frac{\partial}{2\pi} \right)^\alpha f$;
- $(f * g) = \hat{f} \cdot \hat{g}$;
- $(f \cdot g) = \hat{f} * \hat{g}$.

Exercise 5 (3pt). Gaussian Mollifier

Prove that $(\frac{1}{\pi \varepsilon^d})^{\frac{d}{2}} e^{-\frac{|x|^2}{\varepsilon^2}}$ is a mollifier, i.e. that for any measurable function $f$ such that $\exists N \in \mathbb{N}$, $(1 + | \cdot |)^{-N} f( \cdot ) \in L^1(\mathbb{R}^d)$:

$$(\frac{1}{\pi \varepsilon^d})^{\frac{d}{2}} (e^{-\frac{|x|^2}{\varepsilon^2}} * f)(x) \to f(x) \quad (\text{for almost every } x \in \mathbb{R}^d).$$

Exercise 6 (5pt). Compactly supported distributions

Prove that any $T \in \mathcal{D}'(\mathbb{R}^d)$ with $\text{supp}(T)$ compact is tempered (i.e. $T \in \mathcal{S}'(\mathbb{R}^d)$).