Exercise sheet 2

Nonlinear Dispersive PDEs
Sommersemester 2019
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Exercise 1 (6pt). Inequalities

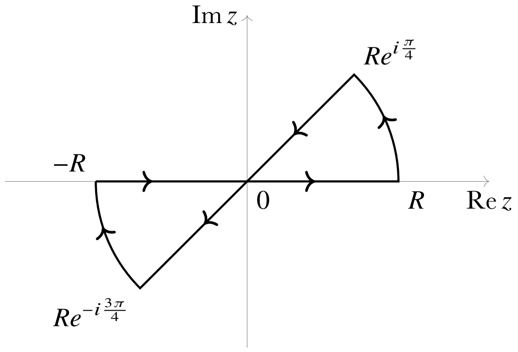
[Justify your answers]

- Let $u \in H^1(\mathbb{R}^3)$, $A \in \dot{H}^1(\mathbb{R}^3)$. Is $A|u|^2 = Au\bar{u} \in L^1(\mathbb{R}^3)$?
- Let $u, v \in L^7(\mathbb{R}^5)$, $w \in H^2(\mathbb{R}^5)$. For which values of p is $u(v * w) \in L^p(\mathbb{R}^5)$?

Exercise 2 (5pt). Fourier transform

Compute the Fourier transform of $e^{-\lambda|x|^2 + i\eta\cdot x} \in \mathscr{S}(\mathbb{R}^d)$, where $\lambda > 0$ and $\eta \in \mathbb{R}^d$.

Optional (10pt). Compute the Fourier transform of $e^{i\lambda x^2 + i\eta x} \in \mathscr{S}'(\mathbb{R})$, $\lambda > 0$, $\eta \in \mathbb{R}$. *Hint:* Even if this is a distribution, it is sufficient to see the Fourier transform integral as one part of the limit $R \to \infty$ of a complex R-dependent contour integral. The contour is



Exercise 3 (7pt). Sobolev functions

Prove that if $u \in \dot{H}^s(\mathbb{R}^d)$, $s < \frac{d}{2}$, then $(-\Delta)^\lambda u \in \dot{H}^{s-2\lambda}(\mathbb{R}^d)$ for all $\lambda > 0$.

[*Hint:* This exercise can be solved studying a linear PDE in $\mathscr{S}'_h$, very similarly to what we did in the class.]

Exercise 4 (5pt). Sobolev norm

Prove that it is possible to write the norm $H^\sigma(\mathbb{R}^d)$, $\sigma \in \mathbb{R}$, of u as the L^2-norm of $f(D)u$, for a suitable pseudodifferential operator $f(D)$. For $\sigma = 1$, prove in addition that

$$\| u\,;\, H^1 \|^2 = \|u\|_2^2 + \tfrac{1}{4\pi^2}\|\nabla u\|_2^2 = \|u\|_2^2 + \tfrac{1}{4\pi^2}\sum_{j=1}^{d}\|\partial_j u\|_2^2\ .$$

Exercise 5 (7pt). Symmetries

Find the Hamiltonian $H(u_t, \alpha_t, i\bar{u}_t, i\bar{\alpha}_t)$ of the nonlinear *system* of PDEs:

$$\text{(S-W)} \qquad \begin{cases} i\partial_t u_t(x) = -\Delta_x u_t(x) + A_t(x)u_t(x) \\ i\partial_t \alpha_t(k) = \omega(k)\alpha_t(k) + \frac{1}{\sqrt{\omega(k)}}\widehat{(|u_t|^2)}(k) \end{cases} ;$$

where $\omega(k) = |k|$, and

$$A_t(x) = \int_{\mathbb{R}^3} \frac{1}{\sqrt{\omega(k)}}\Big(\alpha_t(k)e^{2\pi i k\cdot x} + \bar{\alpha}_t(k)e^{-2\pi i k\cdot x}\Big)\mathrm{d}k \ .$$

The *two* fields are $\varphi_{1,t} = u_t$ and $\varphi_{2,t} = \alpha_t$, and the associated momenta are $\pi_{1,t} = i\bar{u}_t$ and $\pi_{2,t} = i\bar{\alpha}_t$ (A_t instead should be considered as a function of α_t and $\bar{\alpha}_t$, but it may appear in the Hamiltonian). Is (S-W) invariant with respect to $\mathrm{U}(1)$ transformations on u_t? And with respect to $\mathrm{U}(1)$ transformations on α_t?

You may justify the steps in this exercise only formally, as it was done during the lecture.

N.B. The points of the optional exercise are bonus points: they are added to your overall points but they are not counted in the total of available points.