Exercise 1 (5pt). Inequalities II
Let $u \in L^1(\mathbb{R}^3)$, $v \in H^1(\mathbb{R}^3)$, and $w \in L^p(\mathbb{R}^3)$. For which value of $p \in [1, \infty]$, is $\hat{u}(v \ast w) \in L^6(\mathbb{R}^3)$?

Exercise 2 (10pt). Contraction estimate for NLS
Prove Lemma v.18 from the lecture: Let $f$ satisfy (H1), (H2), (H3), let $h$ satisfy (h1) and $g$ satisfy (g1); also, let $r_j = p_j + 1, j = 1, 2$. Then for all $v \in \{(h,g), g, \emptyset\}$, the map $v \mapsto V_v(v)$ is continuous from $X = L^{r_0} \cap L^{r_1}$ to $L^1$ and satisfies: $\forall v_1, v_2 \in X$

$$\|V_v(v_1) - V_v(v_2)\|_1 \leq C \sum_{i,j=1}^2 \|v_1 - v_2\|_{r_j} \|v_i\|_{p_j}.$$

Exercise 3 (15pt). Contractions
Let $X = H^1(\mathbb{R}^d)$, $\mathcal{X}(I) = C^0(I, X)$. Consider the map $A(t_0, u_0), t_0 \in \mathbb{R}, u_0 \in X$, defined as: $\forall u \in \mathcal{X}(I)$

$$[A(t_0, u_0)u](t, x) = e^{i(\tau - t_0)}u_0(x) - i \int_{t_0}^\tau e^{i(t - \tau)}(V \ast u(\tau))(x)u(\tau, x) d\tau,$$

where $V \in L^2(\mathbb{R}^d)$. For any $\rho > 0$, find $T(\rho) > 0$ such that for any $u_0 \in H^1: \|u_0 \ast H^1\| \leq \rho$, then $A(t_0, u_0)$ is a strict contraction on $B(I, 2\rho)$, where $I = [t_0 - T(\rho), t_0 + T(\rho)]$.

More precisely, you should prove the following steps (if you are not able to prove one, you may use it to prove the following ones):

- Prove that the gradient acts on $(f \ast g), f \in L^2$ and $g \in H^1$, as follows: $\nabla (f \ast g) = f \ast (\nabla g)$.

Use this information to deduce that for any $f \in L^2$ and $g, h \in H^1$

$$\| (f \ast g) h \ast H^1 \|^2 = \frac{1}{4\pi^2} \| (f \ast g) h \|^2_2 + \| (f \ast g) h \|^2_2 \leq \frac{1}{2\pi^2} \left( \| (f \ast (\nabla g)) h \|^2_2 + \| (f \ast g) \nabla h \|^2_2 \right) + \| (f \ast g) h \|^2_2.$$

The last bound implies

$$\| (f \ast g) h \ast H^1 \| \leq \frac{1}{\sqrt{2\pi}} \left( \| (f \ast (\nabla g)) h \|^2_2 + \| (f \ast g) \nabla h \|^2_2 \right) + \| (f \ast g) h \|^2_2.$$
• Use the above bound to prove that for any \( t > t_0 \) (for \( t < t_0 \) being analogous) and for any \( u_1, u_2 \in \mathcal{X}(I) \):

\[
|A(t_0, u_0)u_1 - A(t_0, u_0)u_2|_I \leq \frac{1}{\sqrt{2\pi}} \|V\|_2(|u_1|_I + |u_2|_I)(2 + \sqrt{2\pi})T(q)|u_1 - u_2|_I
\]

\[
\leq \frac{4\|V\|_2}{\pi}(\sqrt{2} + \pi)|u_1 - u_2|_I.
\]

• Choose the time \( T(q) \) in the above expression that gives a strict contraction estimate (with contraction constant \( \frac{1}{2} \)). Then check that from this choice it also follows that \( A(t_0, u_0)u \) maps \( B(I, 2q) \) into itself (use the fact that \( |A(t_0, u_0)u|_I \leq |e^{i(\cdot-t_0)}u_0|_I + |A(t_0, u_0)u-A(t_0, u_0)0|_I \).