Exercise sheet 3

Nonlinear Dispersive PDEs
Sommersemester 2019
M. Falconi

Exercise 1 (5pt). Inequalities II

Let $u \in L^1(\mathbb{R}^3)$, $v \in H^1(\mathbb{R}^3)$, and $w \in L^p(\mathbb{R}^3)$. For which value of $p \in [1, \infty]$, is $\hat{u}(v * w) \in L^6(\mathbb{R}^3)$?

Exercise 2 (10pt). Contraction estimate for NLS

Prove Lemma v.18 from the lecture: *Let f satisfy (H1), (H2), (H3), let h satisfy (h1) and g satisfy (g1); also, let* $r_j = p_j + 1$, $j = 1, 2$. *Then for all* $\nu \in \{(h, g), g, \emptyset\}$, *the map* $v \mapsto V_\nu(v)$ *is continuous from* $X = L^{r_0} \cap L^r$ *to* L^1 *and satisfies:* $\forall v_1, v_2 \in X$

$$\|V_\nu(v_1) - V_\nu(v_2)\|_1 \le C \sum_{i,j=1}^{2} \|v_1 - v_2\|_{r_j} \|v_i\|_{r_j}^{p_j} ,$$

$$\|V_\nu(v_1) - V_\nu(v_2)\|_1 \le C \|v_1 - v_2\|_X \sum_{i,j=1}^{2} \|v_i\|_X^{p_j} .$$

Exercise 3 (15pt). Contractions

Let $X = H^1(\mathbb{R}^d)$, $\mathscr{X}(I) = C^0(I, X)$. Consider the map $A(t_0, u_0)$, $t_0 \in \mathbb{R}$, $u_0 \in X$, defined as: $\forall u \in \mathscr{X}(I)$

$$[A(t_0, u_0)u](t, x) = e^{i(t-t_0)} u_0(x) - i \int_{t_0}^{t} e^{i(\tau - t_0)} (V * u(\tau))(x) u(\tau, x) \mathrm{d}\tau ,$$

where $V \in L^2(\mathbb{R}^d)$. For any $\varrho > 0$, find $T(\varrho) > 0$ such that for any $u_0 \in H^1$: $\|u_0\,; H^1\| \le \varrho$, then $A(t_0, u_0)$ is a *strict contraction* on $B(I, 2\varrho)$, where $I = [t_0 - T(\varrho), t_0 + T(\varrho)]$.

More precisely, you should prove the following steps (if you are not able to prove one, you may use it to prove the following ones):

- Prove that the gradient acts on $(f * g)$, $f \in L^2$ and $g \in H^1$, as follows: $\nabla(f * g) = f * (\nabla g)$. Use this information to deduce that for any $f \in L^2$ and $g, h \in H^1$:

$$\|(f*g)h\,; H^1\|^2 = \tfrac{1}{4\pi^2}\|\nabla\,(f*g)h\|_2^2 + \|(f*g)h\|_2^2 \le \tfrac{1}{2\pi^2}\Big(\|(f*(\nabla g))h\|_2^2 + \|(f*g)\nabla h\|_2^2\Big) + \|(f*g)h\|_2^2 .$$

The last bound implies

$$\|(f * g)h\,; H^1\| \le \tfrac{1}{\sqrt{2}\pi}\Big(\|(f * (\nabla g))h\|_2 + \|(f * g)\nabla h\|_2\Big) + \|(f * g)h\|_2 .$$

- Use the above bound to prove that for any $t > t_0$ (for $t < t_0$ being analogous) and for any $u_1, u_2 \in \mathscr{X}(I)$:

$$\begin{aligned} |A(t_0, u_0)u_1 - A(t_0, u_0)u_2|_I &\leq \tfrac{1}{\sqrt{2}\pi}\|V\|_2(|u_1|_I + |u_2|_I)(2 + \sqrt{2}\pi)T(\varrho)|u_1 - u_2|_I \\ &\leq \tfrac{4\varrho}{\pi}(\sqrt{2} + \pi)\|V\|_2 T(\varrho)|u_1 - u_2|_I \ . \end{aligned}$$

- Choose the time $T(\varrho)$ in the above expression that gives a strict contraction estimate (with contraction constant $\frac{1}{2}$). Then check that from this choice it also follows that $A(t_0, u_0)u$ maps $B(I, 2\varrho)$ into itself (use the fact that $|A(t_0, u_0)u|_I \leq |e^{i(\cdot - t_0)}u_0|_I + |A(t_0, u_0)u - A(t_0, u_0)0|_I$).