Exercise sheet 4

Nonlinear Dispersive PDEs
Sommersemester 2019
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Exercise 1 (30pt). Strichartz Estimates

Consider the nonlinear Schrödinger Cauchy problem:

$$(\mathrm{NLS}_d) \qquad \begin{cases} i\partial_t u = -\Delta u + \lambda |u|^{\frac{4}{d}} u \\ u\big|_{t=0} = u_0 \end{cases} .$$

Prove that there exists $c > 0$ such that for all $u_0 \in L^2(\mathbb{R}^d)$, $\|u_0\|_2 \leq c$, (NLS_d) has a unique solution $u \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))$.

The proof can be done in four steps:

(i) [15pt] Using Strichartz and Hölder estimates, prove the inequality:

$$\Big\| \int_0^t e^{i(t-\tau)\Delta} |u(\tau)|^{\frac{4}{d}} u(\tau) \mathrm{d}\tau \Big\|_{L_t^{q_1} L_x^{r_1}} \leq C \|u\|^{\frac{4}{d}+1}_{L_t^{q_2} L_x^{r_2}} ,$$

for any $(q_1, r_1), (q_2, r_2)$ admissible.

(ii) [8pt] Use the mean value theorem to prove the analogous contraction estimate

$$\Big\| \int_0^t e^{i(t-\tau)\Delta} \big(|u(\tau)|^{\frac{4}{d}} u(\tau) - |v(\tau)|^{\frac{4}{d}} v(\tau) \big) \mathrm{d}\tau \Big\|_{L_t^{q_1} L_x^{r_1}} \leq C' \|u - v\|_{L_t^{q_2} L_x^{r_2}} \Big(\|u\|^{\frac{4}{d}}_{L_t^{q_2} L_x^{r_2}} + \|v\|^{\frac{4}{d}}_{L_t^{q_2} L_x^{r_2}} \Big) .$$

(iii) [7pt] Use the above contraction estimate, the first Strichartz inequality, and Banach fixed point theorem to prove that the map

$$[A(0, u_0)](u) = e^{it\Delta} u_0 - i\lambda \int_0^t e^{i(t-\tau)\Delta} |u(\tau)|^{\frac{4}{d}} u(\tau) \mathrm{d}\tau$$

is a contraction on $L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))$, provided $\|u_0\|_2$ is small enough.

(iv) [Bonus: 10pt] Use the contraction estimate to prove uniqueness, on $L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))$, of the solution given by the contraction map. Also, prove that the solution actually belongs to any $L^q(\mathbb{R}, L^r(\mathbb{R}^d))$, with (q, r) admissible.