Exercise 1 (30pt). Strichartz Estimates

Consider the nonlinear Schrödinger Cauchy problem:

\[ \begin{cases} i \partial_t u = -\Delta u + \lambda |u|^4 u \\ u|_{t=0} = u_0 \end{cases} \]

Prove that there exists \( c > 0 \) such that for all \( u_0 \in L^2(\mathbb{R}^d) \), \( \|u_0\|_2 \leq c \), \((\text{NLS}_d)\) has a unique solution \( u \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^d)) \).

The proof can be done in four steps:

(i) [15pt] Using Strichartz and Hölder estimates, prove the inequality:

\[ \| \int_0^t e^{i(t-\tau)\Delta} |u(\tau)|^4 u(\tau) \, d\tau \|_{L^{q_1}_{t} L^{r_1}_{x}} \leq C \|u\|_{L^{q_2}_{t} L^{r_2}_{x}}^{\frac{q_1}{4}+1} , \]

for any \((q_1, r_1), (q_2, r_2)\) admissible.

(ii) [8pt] Use the mean value theorem to prove the analogous contraction estimate

\[ \| \int_0^t e^{i(t-\tau)\Delta} (|u(\tau)|^4 u(\tau) - |v(\tau)|^4 v(\tau)) \, d\tau \|_{L^{q_1}_{t} L^{r_1}_{x}} \leq C' \|u - v\|_{L^{q_2}_{t} L^{r_2}_{x}} \left( \|u\|_{L^{q_2}_{t} L^{r_2}_{x}}^{\frac{q_1}{4}} + \|v\|_{L^{q_2}_{t} L^{r_2}_{x}}^{\frac{q_1}{4}} \right) . \]

(iii) [7pt] Use the above contraction estimate, the first Strichartz inequality, and Banach fixed point theorem to prove that the map

\[ [A(0, u_0)](u) = e^{it\Delta} u_0 - i \lambda \int_0^t e^{i(t-\tau)\Delta} |u(\tau)|^4 u(\tau) \, d\tau \]

is a contraction on \( L^\infty(\mathbb{R}, L^2(\mathbb{R}^d)) \), provided \( \|u_0\|_2 \) is small enough.

(iv) [Bonus: 10pt] Use the contraction estimate to prove uniqueness, on \( L^\infty(\mathbb{R}, L^2(\mathbb{R}^d)) \), of the solution given by the contraction map. Also, prove that the solution actually belongs to any \( L^q(\mathbb{R}, L^r(\mathbb{R}^d)) \), with \((q, r)\) admissible.