Exercise sheet 4

Nonlinear Dispersive PDEs
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Exercise 1 (20pt). Tempered Distributions

Are the following functionals tempered distributions? [Check both *linearity* and *continuity*]

- $T : \mathscr{S}(\mathbb{R}^d) \to \mathbb{C}$,
$$\langle T,f\rangle := \int_{\mathbb{R}^d} |f(x)| \mathrm{d}x \ .$$
- Let $\alpha : \mathbb{R} \to \mathbb{C}$ satisfy $|\alpha(x)| \leq (1+|x|^{16})$ for any $x \in \mathbb{R}$. Then $T_\alpha : \mathscr{S}(\mathbb{R}) \to \mathbb{C}$ is defined by
$$\langle T_\alpha,f\rangle := \int_{\mathbb{R}} \alpha(x) f'''(x) \mathrm{d}x \ .$$
- $T : \mathscr{S}(\mathbb{R}^d) \to [0,+\infty]$,
$$\langle T,f\rangle := \int_{\mathbb{R}^d} e^{|x|^2} |f(x)| \mathrm{d}x \ .$$
- Let $E = \{x \in [0,1]\ ,\ x = \sum_{j=1}^{\infty} \varepsilon_j 3^{-j}\ ,\ \varepsilon_j \in \{0,2\}\}$ be the Cantor set, and $E_{\mathrm{c}} = \mathbb{R} \setminus E$. Also, let $g \in \mathscr{S}(\mathbb{R})$ be fixed. Then $T_{E,g} : \mathscr{S}(\mathbb{R}) \to \mathbb{C}$ is defined by:
$$\langle T_{E,g},f\rangle := \int_{E_{\mathrm{c}}} g(x) f(x) \mathrm{d}x \ .$$
Also, prove that $T_{E,g}$ equals T_g, defined by
$$\langle T_g,f\rangle := \int_{\mathbb{R}^d} g(x) f(x) \mathrm{d}x \ .$$
- $T : \mathscr{S}(\mathbb{R}^d) \to \mathbb{C}$,
$$\langle T,f\rangle := \int_{\mathbb{R}^d} (|D_x|^5 + 4|D_x|^3 + |D_x| + 5) f(x) \mathrm{d}x \ ,$$
where $D_x = \frac{i\nabla}{2\pi}$, and $|D_x|^5 + 4|D_x|^3 + |D_x| + 5$ should be considered as a pseudodifferential operator.

Exercise 2 (10pt). Uncertainty principle

Let $\psi \in \mathscr{S}(\mathbb{R})$, such that $\|\psi\|_2^2 = \int_{\mathbb{R}} |\psi(x)|^2 \mathrm{d}x = 1$. Prove that
$$\|x\psi(x)\|_2^2 \|\xi\hat{\psi}(\xi)\|_2^2 = \int_{\mathbb{R}\times\mathbb{R}} x^2 |\psi(x)|^2 \xi^2 |\hat{\psi}(\xi)|^2 \mathrm{d}x\mathrm{d}\xi \geq \frac{1}{16\pi^2} \ .$$
[*Hint:* $1 = \|\psi\|_2^2 = -\int_{\mathbb{R}} x \frac{\mathrm{d}}{\mathrm{d}x} |\psi(x)|^2 \mathrm{d}x = -\int_{\mathbb{R}} (x\psi'(x)\bar{\psi}(x) + x\bar{\psi}'(x)\psi(x)) \mathrm{d}x$.]