Exercise sheet 5

Nonlinear Dispersive PDEs
Sommersemester 2018
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Exercise 1 (9pt). Duality in spacetime spaces

Prove that for all $(q, r) \in [1, \infty]^2$, if $f \in L_t^q L_x^r$ and $g \in L_t^{q'} L_x^{r'}$ (where $\frac{1}{q'} + \frac{1}{q} = 1 = \frac{1}{r'} + \frac{1}{r}$), then $fg \in L^1(\mathbb{R}_t \times \mathbb{R}_x^d)$. In particular, prove that

$$\left| \int_{\mathbb{R}_t \times \mathbb{R}_x^d} f(t,x) g(t,x) \mathrm{d}t \mathrm{d}x \right| \leq \int_{\mathbb{R}_t \times \mathbb{R}_x^d} |f(t,x) g(t,x)| \mathrm{d}t \mathrm{d}x \leq \|f\|_{L_t^q L_x^r} \|g\|_{L_t^{q'} L_x^{r'}} \ .$$

(*Hint:* Remember the definition of the spacetime norms and apply Hölder's inequality in both time and space in the correct order)

Exercise 2 (21pt). Strichartz estimates

Let $(U(t))_{t \in \mathbb{R}}$ be a family of continuous operators on $L^2(\mathbb{R}^5)$, with norm uniformly bounded w.r.t. t. In addition, suppose that it satisfies for all $t \neq t'$

$$\|U(t) U^*(t') f\|_\infty \leq \frac{1}{|t - t'|^2} \|f\|_1 \ .$$

Prove the following assertions:

- Prove that for all $g \in L_t^4 L_x^8$, $\int_0^t U(t) U^*(\tau) |g(\tau)|^4 \mathrm{d}\tau \in L_t^2 L_x^4$. In other words, prove that

$$\left\| \int_0^t U(t) U^*(\tau) |g(\tau)|^4 \mathrm{d}\tau \right\|_{L_t^2 L_x^4} \leq C \|g\|_{L_t^4 L_x^8}^4 \ .$$

- Find at least one couple $(q, s) \in [1, \infty]^2$ (or prove that there is no such couple) such that for any $g \in L_t^q L_x^s$, $g \geq 0$,

$$\left\| \int_0^t U(t) U^*(\tau) g(\tau)^{\frac{1}{2}} \mathrm{d}\tau \right\|_{L_t^3 L_x^3} < \infty \ .$$

- Let $V \in L_x^{\frac{4}{3}}(\mathbb{R}^5)$, and $f \in L_t^1 L_x^{\frac{4}{3}}$. Is it true that

$$\int_{\mathbb{R}} U^*(t) (V * f(t)) \mathrm{d}t$$

belongs to $L_x^2(\mathbb{R}^5)$?