UNIVERSITÉ DE RENNES 1

Marco Falconi

Centre Henri Lebesgue et IRMAR;
Université de Rennes 1.

Bohr's correspondence principle and classical dressing renormalization in the Nelson model

(Joint work with Zied Ammari)

Porquerolles, June 11th 2015.

Introduction: the renormalized Nelson model

The formal Nelson Hamiltonian.

- We consider two fields in interaction; the first describes bosonic nucleons ($\psi^{\#}(x)$), the second a bosonic meson field ($a^{\#}(k)$).
- The Hilbert space of the theory is $\mathcal{H} = \Gamma_s\big(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\big)$.
- We utilize the ε-dependent form of the CCR: $\boxed{[\psi(x), \psi^*(x')] = \varepsilon\delta(x - x') \ , \ [a(k), a^*(k')] = \varepsilon\delta(k - k')}$.
- Let $\omega(k) = \sqrt{k^2 + m_0^2}$, $M, m_0 > 0$. The formal Nelson operator is $H = H_0 + H_I$,

$$\boxed{H = \int_{\mathbb{R}^3} \psi^*(x)\Big(-\frac{\Delta}{2M} \cancel{+V}\Big)\psi(x)dx + \int_{\mathbb{R}^3} a^*(k)\omega(k)a(k)dk + \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \psi^*(x)\Big(a^*\Big(\frac{e^{-ik\cdot x}}{\sqrt{2\omega}}\Big) + a\Big(\frac{e^{-ik\cdot x}}{\sqrt{2\omega}}\Big)\Big)\psi(x)dx} \ .$$

- **Problem (!):** $\frac{e^{-ik\cdot x}}{\sqrt{2\omega(k)}} \notin L^2_k(\mathbb{R}^3)$. Therefore H is not a densely defined operator on $\mathcal{H}$; it makes sense only as a quadratic form $h(\cdot,\cdot) = \langle \cdot\,, H\cdot \rangle$. In addition, $|h_I(\Psi,\Psi)| \not\lesssim h_0(\Psi,\Psi)$; so there is little hope that h defines a self-adjoint operator as it is.

- To define a regular dynamics, we need to manipulate the quadratic form h , and perform a so-called *renormalization procedure*.

- It is convenient (and a posteriori necessary) to exploit the conservation of the number of nucleons N_1 . For any $\Psi, \Phi \in Q(h)$ and $n_1 \leq n_2 \in \mathbb{N}$:

$$h\Big(\Psi, \mathbb{1}_{[n_1,n_2]}(N_1)\Phi\Big) = h\Big(\mathbb{1}_{[n_1,n_2]}(N_1)\Psi, \mathbb{1}_{[n_1,n_2]}(N_1)\Phi\Big)\;;$$

therefore $\boxed{h = \bigoplus_{n\in\mathbb{N}} h_n}$, with h_n defined on $\mathcal{H}_n = \Big(L^2(\mathbb{R}^3)\Big)^{\otimes_s n} \otimes \Gamma_s\Big(L^2(\mathbb{R}^3)\Big)$; and $\boxed{\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n}$.

Regularized interaction, dressed Hamiltonian.

- Introducing a (smooth) ultraviolet cut off function $\chi_\sigma(k)$ the regularized Nelson operator H_σ becomes essentially self-adjoint on $D(H_0) \cap C_0^\infty(N)$:

$$H_\sigma = \int_{\mathbb{R}^3} \psi^*(x)\Big(-\tfrac{\Delta}{2M}\Big)\psi(x)dx + \int_{\mathbb{R}^3} a^*(k)\omega(k)a(k)dk + \tfrac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \psi^*(x)\Big(a^*\Big(\chi_\sigma \tfrac{e^{-ik\cdot x}}{\sqrt{2\omega}}\Big) + a\Big(\chi_\sigma \tfrac{e^{-ik\cdot x}}{\sqrt{2\omega}}\Big)\Big)\psi(x)dx \,.$$

- It is now possible to introduce an operator, called *dressing* $\boxed{e^{-\frac{i}{\varepsilon}T_\sigma(\sigma_0)}}$ that singles out the part of H that is unbounded from below. The latter turns out to be a scalar quantity (self-energy) $E_\sigma(\sigma_0)$. For any $\sigma_0 \in \mathbb{R}_+$, $\boxed{\lim_{\sigma\to\infty} E_\sigma(\sigma_0) = -\infty}$.

- The dressed Hamiltonian is then:

$$\boxed{\hat{H}_\sigma(\sigma_0) = e^{\frac{i}{\varepsilon}T_\sigma(\sigma_0)} H_\sigma e^{-\frac{i}{\varepsilon}T_\sigma(\sigma_0)} - \varepsilon N_1 E_\sigma(\sigma_0)} \,.$$

Remarks

- The σ_0 dependence is introduced in the dressing generator

$$T_\sigma(\sigma_0) = \int \psi^*(x)\Big(a^*\big(g_\sigma(\sigma_0)e^{-ik\cdot x}\big) + a\big(g_\sigma(\sigma_0)e^{-ik\cdot x}\big)\Big)\psi(x)dx \ ,$$

  through the function:

$$\boxed{g_\sigma(\sigma_0) = (\chi_\sigma - \chi_{\sigma_0})\, g_0 \ , \ g_0 \in L^2(\mathbb{R}^3)} \ .$$

- For any $0 \leq \sigma_0 \leq \sigma \leq \infty$, $T_\sigma(\sigma_0)$ is a self-adjoint operator.

The renormalized Hamiltonian $\sigma \to \infty$.

- We would like to define a self-adjoint operator in the limit $\sigma \to \infty$. In order to do that we have to exploit the parameter $\boxed{\sigma_0}$ and restrict to the $\boxed{n\text{-nucleons sector } \mathcal{H}_n}$.

Theorem (Nelson [1964])

For any $n \in \mathbb{N}$, there exists a $\sigma_0(n, \varepsilon) \sim n\varepsilon$ such that the following statements are true for any $\boxed{\sigma_0 < \sigma \leq \infty}$ *:*

- $\boxed{\hat{H}_\sigma^{(n)}(\sigma_0) \text{ is self-adjoint with domain } \hat{D}_\sigma^{(n)} \subset Q(H_0^{(n)}) \subset \mathcal{H}_n}$ *. It is uniquely associated with the symmetric form* $\hat{h}_\sigma^{(n)}(\cdot, \cdot)$ *defined by* $\hat{h}_\sigma(\cdot, \cdot) = \bigoplus_{n \in \mathbb{N}} \hat{h}_\sigma^{(n)}(\cdot, \cdot)$.
- $\boxed{\lim_{\sigma \to \infty} \hat{H}_\sigma^{(n)}(\sigma_0) = \hat{H}_\infty^{(n)}(\sigma_0)}$ *in the norm resolvent sense.*
- $\boxed{\lim_{\sigma \to \infty} e^{-i\frac{t}{\varepsilon}\hat{H}_\sigma^{(n)}(\sigma_0)} = e^{-i\frac{t}{\varepsilon}\hat{H}_\infty^{(n)}(\sigma_0)}}$ *in the strong sense.*

- How to extend $\hat{H}_{\infty}^{(n)}(\sigma_0)$ to an operator on the whole $\mathcal{H}$?
- We choose to "cut" the dynamics to be relevant only up to some $\mathfrak{N}$-nucleons sector; where $\mathfrak{N}(\sigma_0, \varepsilon)$ depends on a fixed $\sigma_0 \in \mathbb{R}_+$ (that will be chosen by other considerations).

Definition (Renormalized Hamiltonians)

Let $\sigma_0 \in \mathbb{R}_+$. Then there exist a $\mathfrak{N}(\sigma_0, \varepsilon)$ such that the operators $\hat{H}_{ren}(\sigma_0)$ and $H_{ren}(\sigma_0)$ defined by

$$\hat{H}_{ren}(\sigma_0)\big|_{\mathcal{H}_n} = \begin{cases} \hat{H}_{\infty}^{(n)}(\sigma_0) & \text{if } n \leq \mathfrak{N}(\varepsilon, \sigma_0) \\ 0 & \text{if } n > \mathfrak{N}(\varepsilon, \sigma_0) \end{cases} ,$$

$$H_{ren}(\sigma_0) = e^{-\frac{i}{\varepsilon} T_{\infty}(\sigma_0)} \hat{H}_{ren}(\sigma_0) e^{\frac{i}{\varepsilon} T_{\infty}(\sigma_0)} .$$

are self-adjoint on $\mathcal{H}$.

Classical dressing renormalization

The classical equations.

(S-KG[Y])

$$\begin{cases} i\partial_t u = -\frac{\Delta}{2M} u + Au \\ (\Box + m_0^2)A = -|u|^2 \end{cases}$$

- The system (S-KG[Y]) is globally well-posed on the energy space $H^1(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3)$ (Bachelot [1984]).
- Setting $A = \sqrt{2}\mathrm{Re}\mathcal{F}^{-1}(\omega^{-1/2}\alpha)$, we obtain the equivalent equation for the variables $\boxed{z = (u, \alpha)}$:

(S-KG$_\alpha$[Y])

$$\boxed{\begin{cases} i\partial_t u = -\frac{\Delta}{2M} u + Au \\ i\partial_t \alpha = \omega\alpha + \frac{1}{\sqrt{2\omega}}\mathcal{F}\big(|u|^2\big) \end{cases}} ;$$

globally well posed on the energy space $H^1 \oplus \mathcal{F}H^{1/2} \subset L^2 \oplus L^2$.

- The (S-KG$_\alpha$[Y]) equation can be seen as the Hamilton-Jacobi equation for the following phase space energy functional $\mathscr{E} : D(\mathscr{E}) \to \mathbb{R}$, defined by:

$$\boxed{\mathscr{E}(u,\alpha) := \Big\langle u, \big(-\tfrac{\Delta}{2M}\big)u\Big\rangle_2 + \Big\langle \alpha, \omega\alpha\Big\rangle_2 + \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \frac{1}{\sqrt{2\omega(k)}}\Big(\bar{\alpha}(k)e^{-ik\cdot x} + \alpha(k)e^{ik\cdot x}\Big)\big|u(x)\big|^2 dxdk} \,.$$

- $D(\mathscr{E}) \supseteq H^1 \oplus \mathcal{F}H^{1/2}$ is dense in the phase space $L^2 \oplus L^2$.

- $\boxed{h(\cdot,\cdot) = \langle \cdot, (\mathscr{E})^{Wick} \cdot \rangle}$.

- We denote by $\mathbf{E}(\cdot) : \mathbb{R} \times (H^1 \oplus \mathcal{F}H^{1/2}) \to H^1 \oplus \mathcal{F}H^{1/2}$ the Hamiltonian flow associated to $\mathscr{E}$; i.e. $\boxed{\mathbf{E}(t)(u,\alpha)}$ is the solution to the (S-KG$_\alpha$[Y]) Cauchy problem with initial datum $(u,\alpha) \in H^1 \oplus \mathcal{F}H^{1/2}$.

The classical dressing transformation.

Let $g_\infty(\sigma_0) \in \mathcal{F}H^{1/2}$. The dressing generator $\mathscr{D}_{g_\infty} : L^2 \oplus L^2 \to \mathbb{R}$ is defined by:

$$\boxed{\mathscr{D}_{g_\infty}(u,\alpha) := \int_{\mathbb{R}^6} \Big(g_\infty(k)\bar{\alpha}(k)e^{-ik\cdot x} + \bar{g}_\infty(k)\alpha(k)e^{ik\cdot x} \Big) |u(x)|^2 dxdk} \, .$$

Proposition

- *Let $g_\infty(\sigma_0) \in \mathcal{F}H^{1/2}$. Then the classical dressing $\mathbf{D}_{g_\infty}(\cdot) : \mathbb{R} \times (H^1 \oplus \mathcal{F}H^{1/2}) \to H^1 \oplus \mathcal{F}H^{1/2}$ is globally well-defined.*
- *If, in addition, g_∞ has fixed parity, then $\mathbf{D}_{g_\infty}(\theta)$ has the explicit form:*

$$\boxed{\mathbf{D}_{g_\infty}(\theta)(u(x),\alpha(k)) = \Big(u(x)e^{-i\theta A_{g_\infty}(x)} \, , \, \alpha(k) - i\theta(2\pi)^{3/2} g_\infty(k)\mathcal{F}(|u|^2)(k) \Big)} \; ;$$

where $A_{g_\infty} = 2(2\pi)^{3/2}\mathrm{Re}\mathcal{F}(g_\infty\bar{\alpha})$.

Dressed energy.

Since $\mathbf{D}_{g_\infty}(\theta)$ maps the energy space $H^1 \oplus \mathcal{F}H^{1/2}$ into itself, we can define the dressed energy

$$\boxed{\hat{\mathscr{E}}(u,\alpha) = \mathscr{E} \circ \mathbf{D}_{g_\infty}(1)(u,\alpha)}\;.$$

We can write $\hat{\mathscr{E}}$ explicitly as

$$\begin{aligned}\hat{\mathscr{E}}(u,\alpha) :=& \left\langle u, \left(-\tfrac{\Delta}{2M}\right)u\right\rangle_2 + \left\langle \alpha, \omega\alpha\right\rangle_2 + \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^6} \frac{\chi_{\sigma_0}(k)}{\sqrt{2\omega(k)}}\Big(\bar{\alpha}(k)e^{-ik\cdot x} + \alpha(k)e^{ik\cdot x}\Big)|u(x)|^2 dxdk \\ &+ \frac{1}{2M}\int_{\mathbb{R}^9}\Big(r_\infty(k)\bar{\alpha}(k)e^{-ik\cdot x} + \bar{r}_\infty(k)\alpha(k)e^{ik\cdot x}\Big)\Big(r_\infty(l)\bar{\alpha}(l)e^{-il\cdot x} + \bar{r}_\infty(l)\alpha(l)e^{il\cdot x}\Big)|u(x)|^2 dxdkdl \\ &- \frac{2}{M}\mathrm{Re}\int_{\mathbb{R}^6} r_\infty(k)\bar{\alpha}(k)e^{-ik\cdot x}\bar{u}(x)D_x u(x)dxdk + \frac{1}{2}\int_{\mathbb{R}^6} V_\infty(x-y)|u(x)|^2|u(y)|^2 dxdy\;;\end{aligned}$$

where $V_\infty(x) = 2\mathrm{Re}\int_{\mathbb{R}^3}\omega(k)|g_\infty(k)|^2 e^{-ik\cdot x}dk - 4\mathrm{Im}\int_{\mathbb{R}^3}\frac{\bar{g}_\infty(k)}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(k)}}e^{-ik\cdot x}dk$; $r_\infty(k) = -ikg_\infty(k)$.

Remarks

- We had the freedom to choose $g_\infty(\sigma_0)$ in a family $\bigl(g_\infty(\sigma_0)\bigr)_{\sigma_0\in\mathbb{R}_+} \subset \mathcal{F}H^{1/2}$.

- To simplify the form of $\hat{\mathscr{E}}$, it is convenient to express this freedom defining $g_\infty(\sigma_0) = (1-\chi_{\sigma_0})g_0$, where $g_0 \in \mathcal{F}H^{1/2}$ cancels the interaction term of $\mathscr{E}$.

- It follows that $(\chi_{\sigma_0})_{\sigma_0\in\mathbb{R}_+} \subset L^\infty \cap \mathcal{F}H^{-1/2}$.

- $\chi_{\sigma_0}(k) = 0$ is a possible choice.

Wick quantization of $\hat{\mathscr{E}}$.

Theorem (Ammari and F. [2015])

- *Let χ_{σ_0} be the smooth ultraviolet cut off function introduced before. Then for any $\Psi_{\leq\mathfrak{N}}, \Phi_{\leq\mathfrak{N}} \in Q\big(\hat{H}_{ren}(\sigma_0)\big) \cap 1_{[0,\mathfrak{N}(\sigma_0,\varepsilon)]}(N_1)\mathcal{H}$:*

$$\boxed{\Big\langle \Psi_{\leq\mathfrak{N}}, \big(\mathscr{E} \circ \mathbf{D}_{g_\infty}(1)\big)^{Wick} \Phi_{\leq\mathfrak{N}} \Big\rangle = \Big\langle \Psi_{\leq\mathfrak{N}}, \hat{H}_{ren}(\sigma_0)\Phi_{\leq\mathfrak{N}} \Big\rangle}\ .$$

  In other words, the Wick quantization of $\hat{\mathscr{E}}(\sigma_0)$ yields the quadratic form uniquely associated with $\hat{H}_{ren}(\sigma_0)$ (when restricted to $1_{[0,\mathfrak{N}(\sigma_0,\varepsilon)]}(N_1)\mathcal{H}$).

- *$\hat{\mathscr{E}}$ generates a globally well-posed flow $\hat{\mathbf{E}}$ on the energy space $H^1 \oplus \mathcal{F}H^{1/2}$; in addition the following identity holds:*

$$\boxed{\mathbf{E}(t) = \mathbf{D}_{g_\infty}(1) \circ \hat{\mathbf{E}}(t) \circ \mathbf{D}_{g_\infty}(-1)}\ .$$

- Is it possible to give a classical meaning to the parameter σ_0, i.e. to the relevant number of nucleons $\mathfrak{N}(\sigma_0, \varepsilon)$ of the quantum theory?

- Since $(\mathscr{D}_{g_\infty})^{Wick} = T_\infty(\sigma_0)$, the latter being the quantum dressing generator, the relation between dressed and undressed classical flows corresponds to the relation

$$e^{-i\frac{t}{\varepsilon}H_{ren}(\sigma_0)} = e^{-\frac{i}{\varepsilon}T_\infty(\sigma_0)}e^{-i\frac{t}{\varepsilon}\hat{H}_{ren}(\sigma_0)}e^{\frac{i}{\varepsilon}T_\infty(\sigma_0)}$$

  between dressed and undressed quantum evolutions. So we expect $H_{ren}(\sigma_0)$ to be indeed the correct quantization of the S-KG system described by $\mathscr{E}$.

The classical motivation of $\mathfrak{N}(\sigma_0, \varepsilon)$: $\mathscr{E}$-boundedness from below.

The boundedness from below of the classical energy imposes a limitation on the number of quantum nucleons, thus fixing $\mathfrak{N}(\sigma_0, \varepsilon)$, and therefore the suitable σ_0:

- $$\inf_{(u,\alpha)\in D(\mathscr{E})} \mathscr{E}(u,\alpha) = -\infty \; .$$
- $$\boxed{\inf_{\substack{(u,\alpha)\in D(\mathscr{E}) \\ \|u\|_2 \leq \sqrt{\mathfrak{C}}}} \mathscr{E}(u,\alpha) > -\infty} \; .$$
- The evolution $\boxed{\mathbf{E}(t) \text{ preserves } \|u\|_2}$; therefore it is natural to restrict the phase space to the fixed *u*-norm configurations.
- In other words, we are interested in $\boxed{\text{classical probability distributions that are supported in the ball } B_u(0, \sqrt{\mathfrak{C}})}$

- Which are the families of quantum states that are associated (by a classical limit procedure) to $B_u(0, \sqrt{\mathfrak{C}})$-supported classical probability distributions?

- These quantum states should constitute the “relavant” part of the quantum Hilbert space, for they are associated with a bounded from below classical energy.

- *Caveat.* The second quantization procedure may introduce, in a theory where the number of particles is conserved, additional singularity: for some sets of quantum states the energy may be unbounded from below even if the corresponding classical one is bounded.

Proposition (Ammari and Nier [2011])

Let $(\lambda_i)_{i\in\mathbb{N}} \in l^1$, $\lambda_i > 0\ \forall i \in \mathbb{N}$; *and* $(\Psi_i)_{i\in\mathbb{N}} \subset \mathcal{H}$, *such that* $\forall i \in \mathbb{N}$, $\boxed{\mathcal{H}_n \ni \Psi_{i,n} = 0\ \forall n > [\mathfrak{C}/\varepsilon]}$, $\mathfrak{C} > 0$. *Then a normal quantum state* $\varrho_\varepsilon \in \mathcal{L}^1(\mathcal{H})$ *has at most* $[\mathfrak{C}/\varepsilon]$ *nucleons if it can be written as:*

$$\varrho_\varepsilon = \sum_{i\in\mathbb{N}} \lambda_i |\Psi_i\rangle\langle\Psi_i| \; .$$

The classical phase space distributions, if any, corresponding to a family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ *of states with at most* $[\mathfrak{C}/\varepsilon]$ *nucleons are supported on* $B_u(0, \sqrt{\mathfrak{C}})$.

- In addition, on these states the additional unboundedness introduced by allowing an arbitrary number of particles in second quantization is avoided (since the number of nucleons is truncated).
- These states are strong candidates to be the relevant quantum states of the theory .

- To sum up, it seems natural to consider as relevant quantum states, corresponding to the restricted phase space $B_u(0, \sqrt{\mathfrak{C}})$ and the quantization procedure, the ones with at most $[\mathfrak{C}/\varepsilon]$ nucleons .

- We exploit the freedom in choosing σ_0 to have a non-trivial dynamics on the relevant quantum states , setting $\mathfrak{N}(\sigma_0, \varepsilon) = [\mathfrak{C}/\varepsilon]$.

Bohr's correspondence principle

$\text{Quantum}_{\varepsilon>0} \longrightarrow \text{Classical}_{\varepsilon=0}$ correspondence in the Nelson model.

- $$\Gamma_s(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)) = \mathcal{H} \quad \longrightarrow \quad L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$$
  Fock-Cook representation of the CCR → Classical phase space

- $$\varrho_\varepsilon \in \mathcal{L}^1(\mathcal{H}),\ \varrho_\varepsilon \geq 0,\ \mathrm{Tr}\,\varrho_\varepsilon = 1 \quad \longrightarrow \quad \mu \in \mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})\big) \subset P\big(L^2 \oplus L^2\big)$$
  ε-dependent normal quantum state → Probability (Wigner) measures on the classical phase space

- $$F := f(\psi, a) \in \mathrm{ClOp}(\mathcal{H}),\ F = F^* \quad \longrightarrow \quad \mathscr{F}(u,\alpha) : D(\mathscr{F}) \to \mathbb{R}$$
  Quantum observables affiliated to the CCR algebra → Classical observables (phase-space functionals)

- $$e^{-i\frac{t}{\varepsilon}H_{ren}},\ e^{-i\frac{t}{\varepsilon}\hat{H}_{ren}} \quad \longrightarrow \quad \mathbf{E}(t),\ \hat{\mathbf{E}}(t)$$
  Quantum evolutions → Classical evolutions

- $$e^{-i\frac{\theta}{\varepsilon}T_\infty(\sigma_0)} \quad \longrightarrow \quad \mathbf{D}_{g_\infty}(\theta)$$
  Quantum dressing → Classical dressing

Bohr's correspondence on states.

- The quantum states depend on ε, so it is convenient to group them in families $\boxed{(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \subset \mathcal{L}^1(\mathcal{H})}$.

- A classical state $\mu \in P\big(L^2 \oplus L^2\big)$ is a Wigner measure associated to a family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$, in symbols $\boxed{\mu \in \mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})\big)}$, if there exists $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\bar{\varepsilon})$, $\lim_{k\to\infty} \varepsilon_k = 0$, such that

$$\boxed{\lim_{k\to\infty} \mathrm{Tr}\big[\varrho_{\varepsilon_k} W(\xi)\big] = \int_{L^2\oplus L^2} e^{i\sqrt{2}\mathrm{Re}\langle \xi, z\rangle_{L^2\oplus L^2}} d\mu(z) \ , \ \forall \xi \in L^2 \oplus L^2} \ .$$

- If the equation above is satisfied, we may also say that the classical measure μ corresponds (i.e. it is uniquely associated) to the sequence $(\varrho_{\varepsilon_k})_{k\in\mathbb{N}}$; in symbols $\boxed{\varrho_{\varepsilon_k} \to \mu}$.

Regularity of quantum states.

- We make the following assumptions on the (initial-time) families of quantum states:

(A_0) $$\boxed{\exists \mathfrak{C} > 0\,,\ \forall \varepsilon \in (0,\bar{\varepsilon})\,,\ \forall k \in \mathbb{N}\,,\ \mathrm{Tr}[\varrho_\varepsilon N_1^k] \leq \mathfrak{C}^k}\ ;$$

(A_ρ) $$\boxed{\exists C > 0\,,\ \forall \varepsilon \in (0,\bar{\varepsilon})\,,\ \mathrm{Tr}[\varrho_\varepsilon (N + U_\infty^* H_0 U_\infty)] \leq C}\ ;$$

- Assumption (A_0) is equivalent to choosing each ϱ_ε with at most $[\mathfrak{C}/\varepsilon]$ nucleons.

- These assumptions are more than sufficient to ensure the existence of at least one classical measure associated to the family:

Proposition (Ammari and Nier [2008])

Under Assumptions (A_0) and (A_ρ),

$$\mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})\big) \neq \emptyset\ .$$

Evolution of Wigner measures: recovering the classical dynamics.

Theorem (Ammari and F. [2015])

Let $\mathbf{E} : \mathbb{R} \times \mathcal{D} \to \mathcal{D}$ *be the S-KG flow associated to* $\mathscr{E}$. *Let* $(\varrho_\varepsilon)_{\varepsilon \in (0,\bar{\varepsilon})}$ *be a family of normal states in* $\Gamma_s\big(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\big)$ *that satisfies Assumptions* A_0 *and* A_ρ.

Then there exists a $\sigma_0 \in \mathbb{R}_+$ *such that the dynamics* $e^{-i\frac{t}{\varepsilon}H_{ren}(\sigma_0)}$ *is non-trivial on every relevant sector with fixed nucleons;* $\mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})\big) \neq \emptyset$ *; and for any* $t \in \mathbb{R}$

$$\mathcal{M}\Big(e^{-i\frac{t}{\varepsilon}H_{ren}(\sigma_0)}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_{ren}(\sigma_0)}, \varepsilon \in (0,\bar{\varepsilon})\Big) = \Big\{\mathbf{E}(t)_{\#}\mu \,,\, \mu \in \mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})\big)\Big\} .$$

More precisely,

$$\boxed{\varrho_{\varepsilon_k} \to \mu \Leftrightarrow \Big(\forall t \in \mathbb{R} \,,\; e^{-i\frac{t}{\varepsilon}H_{ren}(\sigma_0)}\varrho_{\varepsilon_k} e^{i\frac{t}{\varepsilon}H_{ren}(\sigma_0)} \to \mathbf{E}(t)_{\#}\mu\Big)} \,.$$

Remarks

- Despite the renormalization procedure (and the dressing), the Hamiltonian H_{ren} still reproduces the classical behavior of the S-KG system, in the limit of small ε.
- The classical undressed evolution $\mathbf{E}(t)$ is independent of σ_0, even if it is obtained as a limit of $e^{-i\frac{t}{\varepsilon}H_{ren}(\sigma_0)}$.
- A straightforward application of the theorem yields the convergence of (suitably regular) observables' evaluation in the state:

$$\varrho_{\varepsilon_k} \to \mu \Rightarrow \Bigl(\forall t \in \mathbb{R}\,,\ \lim_{k\to\infty} \mathrm{Tr}\Bigl[e^{-i\frac{t}{\varepsilon}H_{ren}(\sigma_0)}\varrho_{\varepsilon_k} e^{i\frac{t}{\varepsilon}H_{ren}(\sigma_0)}(\mathscr{A})^{Wick}\Bigr] = \int_{L^2\oplus L^2} \mathscr{A}(z)d\bigl(\mathbf{E}(t)_{\#}\mu\bigr)(z)\Bigr)\,.$$

- Bohr's correspondence principle is thus rigorously proved in the system for (a large class of) quantum states, observables, and their respective time-evolution.

Outline of the proof I: dressed dynamics and dressing.

- The result is quite difficult to be proved directly, for the generator $H_{ren}(\sigma_0)$ has no known explicit form.

- We prove the following two results, concerning the classical limit of $e^{-i\frac{t}{\varepsilon}\hat{H}_{ren}(\sigma_0)}$ and $e^{-i\frac{\theta}{\varepsilon}T_\infty(\sigma_0)}$:

$$\boxed{\varrho_{\varepsilon_k} \to \mu \Leftrightarrow \left(\forall t \in \mathbb{R}\ ,\ e^{-i\frac{t}{\varepsilon}\hat{H}_{ren}(\sigma_0)}\varrho_{\varepsilon_k} e^{i\frac{t}{\varepsilon}\hat{H}_{ren}(\sigma_0)} \to \hat{\mathbf{E}}(t)_{\#}\mu\right)}\ ;$$

$$\boxed{\varrho_{\varepsilon_k} \to \mu \Leftrightarrow \left(\forall \theta \in \mathbb{R}\ ,\ \forall \sigma_0 \in \mathbb{R}_+\ ,\ e^{-i\frac{\theta}{\varepsilon}T_\infty(\sigma_0)}\varrho_\varepsilon e^{i\frac{\theta}{\varepsilon}T_\infty(\sigma_0)} \to \mathbf{D}_{g_\infty}(\theta)_{\#}\mu\right)}\ .$$

- These two results are sufficient, in combination with the classical relation $\boxed{\mathbf{E}(t) = \mathbf{D}_{g_\infty}(1) \circ \hat{\mathbf{E}}(t) \circ \mathbf{D}_{g_\infty}(-1)}$, to prove the theorem.

- $e^{-i\frac{t}{\varepsilon}H_{ren}(\sigma_0)}\varrho_{\varepsilon_k}e^{i\frac{t}{\varepsilon}H_{ren}(\sigma_0)} = e^{-\frac{i}{\varepsilon}T_\infty(\sigma_0)}e^{-i\frac{t}{\varepsilon}\hat{H}_{ren}(\sigma_0)}e^{\frac{i}{\varepsilon}T_\infty(\sigma_0)}\varrho_{\varepsilon_k}\, e^{-\frac{i}{\varepsilon}T_\infty(\sigma_0)}e^{i\frac{t}{\varepsilon}\hat{H}_{ren}(\sigma_0)}e^{\frac{i}{\varepsilon}T_\infty(\sigma_0)}$.
- Hence we obtain:

$$\varrho_{\varepsilon_k} \to \mu \Leftrightarrow \left(\forall t \in \mathbb{R}\,,\; e^{-i\frac{t}{\varepsilon}H_{ren}(\sigma_0)}\varrho_{\varepsilon_k}e^{i\frac{t}{\varepsilon}H_{ren}(\sigma_0)} \to [\mathbf{D}_{g_\infty}(1)\circ\hat{\mathbf{E}}(t)\circ\mathbf{D}_{g_\infty}(-1)]_{\#}\mu\right).$$

Outline of the proof II: classical limit of the dressed dynamics.

$$\varrho_{\varepsilon_k} \to \mu \Leftrightarrow \left(\forall t \in \mathbb{R} \, , \; e^{-i\frac{t}{\varepsilon}\hat{H}_{ren}(\sigma_0)} \varrho_{\varepsilon_k} e^{i\frac{t}{\varepsilon}\hat{H}_{ren}(\sigma_0)} \to \hat{\mathbf{E}}(t)_{\#}\mu \right)$$

- To prove the above convergence, we are able to adopt the strategy already used for the Nelson model with cut off (Ammari and F. [2014]).
- However the dynamics is, in this case, much more singular: $\hat{H}_{ren}$ can be explicitly defined only as a quadratic form.
- The starting point is to justify the following integral formula for the quantum dynamics:

$$\boxed{\mathrm{Tr}\Big[\tilde{\varrho}_{\varepsilon_k}(t) W(\xi)\Big] = \mathrm{Tr}\Big[\varrho_{\varepsilon_k} W(\xi)\Big] + \frac{i}{\varepsilon_k}\int_0^t \mathrm{Tr}\Big[\varrho_{\varepsilon_k}(s)[\hat{H}_{ren,I}, W(\tilde{\xi}_s)]\Big] ds}$$

- Taking suitably the limit $k \to \infty$ in the above integral formula, we prove that the corresponding classical measure (in interaction picture) $\tilde{\mu}_t$ satisfies the following transport equation (in a weak sense):

$$\boxed{\partial_t \tilde{\mu}_t + \nabla^T \Big(\mathbf{V}(t) \tilde{\mu}_t \Big) = 0} \; ;$$

with the velocity vector field $\mathbf{V}(t)(z) = -i\mathbf{E}_0(-t) \circ \partial_{\bar{z}} \Big(\hat{\mathscr{E}} - \mathscr{E}_0 \Big) \circ \mathbf{E}_0(t)(z)$.

- $\boxed{\mu_t = \hat{\mathbf{E}}(t)_{\#}\mu_0}$ yields a solution of the above transport equation, setting $\tilde{\mu}_t = \mathbf{E}_0(-t)_{\#}\mu_t$.

- In addition, the map $\tilde{\mu}_t$ is sufficiently regular to ensure that such solution is unique (this regularity is given by the assumptions on the quantum states).

Thank you for the attention

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