# Marco Falconi

*Politecnico di Milano*


# A Quantum Detour: regularizing CED by means of QED

(Joint work with Z. Ammari, F. Hiroshima – arXiv:2202.05015

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# Electrodynamics of nonrelativistic charges

# Classical Electrodynamics

# Classical charged particles interacting with the EM field

- Newton–Maxwell Equations:

$$
\text{(N-M)} \qquad \begin{cases}
\dot{\mathbf{q}}_j = \frac{\mathbf{p}_j}{m_j} \\
\dot{\mathbf{p}}_j = m_j(\varrho_j * \mathbf{E})(\mathbf{q}_j) + \mathbf{p}_j \times (\varrho_j * \mathbf{B})(\mathbf{q}_j) - \nabla_j V(\mathbf{q}) \\
\\
\partial_t \mathbf{B}(\cdot) + \nabla \times \mathbf{E}(\cdot) = 0 \\
\partial_t \mathbf{E}(\cdot) - \nabla \times \mathbf{B}(\cdot) = -\sum_j \frac{\mathbf{p}_j}{m_j} \varrho_j(\cdot - \mathbf{q}_j) \\
\\
\nabla \cdot \mathbf{E}(\cdot) = \sum_j \varrho_j(\cdot - \mathbf{q}_j) \\
\nabla \cdot \mathbf{B}(\cdot) = 0
\end{cases}
$$

# Folklore: Disasters with (Almost) Point Charges

- Point Charges:

$$\varrho_{\forall j} = e_j \delta \quad \Longrightarrow \quad \lightning$$

(electrostatic energy unbounded from below, atomic collapse by radiation)

- Charges with a small radius: [1]

$$\varrho_{\forall j} = e_j \mathbb{1}_{\left\{|\cdot| < \frac{2e_j^2}{3m_j}\right\}} \quad \Longrightarrow \quad \lightning$$

(existence of runaway and non-causal solutions)

[1] E.J. Moniz, D.H. Sharp, *Phys. Rev. D* **15**(10), 1977.

# Well-Posedness

- Global Well-Posedness ($V \in \mathscr{C}^2_{\mathrm{b}}$):

  $\varrho_{\forall j}$ "regular enough" $\implies$ GWP on suitable Sobolev spaces for $\mathbf{E}$ and $\mathbf{B}$:

  $\varrho_{\forall j} \in H^1 \implies$ GWP on the space with $\mathbf{E} \in (H^{\frac{1}{2}})^{\times 3}$ and $\mathbf{B} \in (H^{\frac{1}{2}})^{\times 3}$

# Quantum Electrodynamics

# Quantized charged particles interacting with the Quantum EM field (Coulomb Gauge)

- Pauli-Fierz Hamiltonian:

$$\hat{H}_\hbar = \sum_{j=1}^{n} \frac{1}{2m_j} \big(\hat{p}_j - A_j(\hat{q}_j, \hat{a})\big)^2 + V(\hat{q}) + \hat{H}_{\mathrm{f}} \,,$$

$$A_j(\hat{q}_j, \hat{a}) = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} \frac{\epsilon_\lambda(k)}{\sqrt{2|k|}} \Big( \overline{\mathscr{F}\varrho_j}(k)\, \hat{a}_\lambda(k) e^{2\pi i k\cdot \hat{q}_j} + \mathscr{F}\varrho_j(k)\, \hat{a}^*_\lambda(k) e^{-2\pi i k\cdot \hat{q}_j} \Big) \mathrm{d}k \,,$$

$$\hat{H}_{\mathrm{f}} = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} |k| \hat{a}^*_\lambda(k) \hat{a}_\lambda(k) \mathrm{d}k \,,$$

$$[\hat{q}_j, \hat{p}_k] = i\hbar \delta_{jk} \,,\ [\hat{a}_\lambda(k), \hat{a}^*_\mu(p)] = \hbar \delta_{\lambda\mu} \delta(k-p) \,.$$

- Quantum Dynamics:

$$\gamma_\hbar(t) = e^{-i\frac{t}{\hbar}\hat{H}_\hbar} \gamma_\hbar e^{i\frac{t}{\hbar}\hat{H}_\hbar} \,.$$

# Well-Posedness

- Global Well-Posedness ($V \in \mathscr{C}^2_{\mathrm{b}}$):

$$\varrho_{\forall j} \in \dot{H}^{-1} \cap \dot{H}^{\frac{1}{2}} \implies \hat{H}_{\hbar} \text{ is self-adjoint on } D(\hat{p}^2) \cap D(\hat{H}_{\mathrm{f}}) \ .$$

## Remarks

- More singular $V$s are allowed (*e.g.* Coulomb)
- Folklore is that point charges shall be admissible, however it is still mathematically an open problem (a renormalization is required)
- Atoms are stable, and no runaway or non-causal solutions are present

# Bohr's Correspondence and Quantum Driven Classical Trajectories

# Q-Driven Classical Trajectories – Part 1

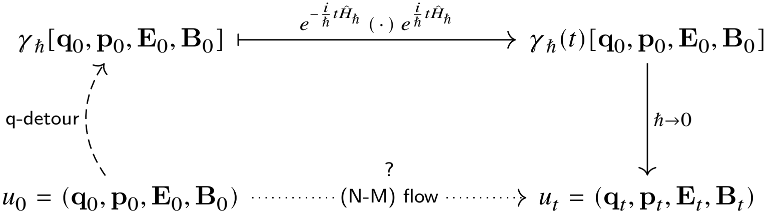
# Quantum Driven Classical GWP

## Theorem 1 (Z. Ammari, MF, F. Hiroshima 2022)

$\varrho_{\forall j} \in \dot{H}^{-1} \cap \dot{H}^{1-\sigma} \Longrightarrow$ (N-M) *GWP on the space s.t.* $\mathbf{E} \in (H^{\sigma})^{\times 3}$ *and* $\mathbf{B} \in (H^{\sigma})^{\times 3}$

$$\boxed{0 \leq \sigma \leq \frac{1}{2}}$$

# Schematic proof of Theorem 1: Taking a Quantum Detour



# Bohr's Correspondence Principle in QED

# Semiclassical q-states

- Wigner Measures:
  - Quantum states:

$$\gamma_\hbar \in \mathfrak{S}^1_+\left(L^2(\mathbb{R}^{3n}) \otimes \Gamma_s(L^2(\mathbb{R}^3, \mathbb{C}^2))\right)$$

  - Classical states:

$$\mu \in \mathscr{P}\Big(\mathbb{R}^{3n} \oplus L^2(\mathbb{R}^3, \mathbb{C}^2)\Big) \mapsto \mu \in \mathscr{P}\Big(\mathbb{R}^{3n} \oplus (L^2(\mathbb{R}^3))^{\times 3} \oplus (L^2(\mathbb{R}^3))^{\times 3}\Big)$$

$$u_\alpha = (\mathbf{q}, \mathbf{p}, (\alpha_1, \alpha_2)) \qquad\qquad u = (\mathbf{q}, \mathbf{p}, \mathbf{E}, \mathbf{B})$$

  - Quantum $\longrightarrow$ Classical (Wigner measure):

$$\gamma_\hbar \underset{\hbar \to 0}{\longrightarrow} \mathrm{d}\mu(u) \iff \mu \text{ is the Wigner measure of } \gamma_\hbar$$

- Noncommutative q-Fourier Transform:

$$\hat{\gamma}_\hbar(u_\alpha) = \mathrm{Tr}\Big(\gamma_\hbar W_\hbar(u_\alpha)\Big) = \mathrm{Tr}\Big(\gamma_\hbar e^{i(\pi(\mathbf{p}\cdot\hat{q}+\mathbf{q}\cdot\hat{p})+\frac{1}{\sqrt{2}}(\hat{a}_1(\alpha_1)+\hat{a}_2(\alpha_2)+\hat{a}_1^*(\alpha_1)+\hat{a}_2(\alpha_2)))}\Big)$$

- Fourier transform of $\mu$:

$$\hat{\mu}(u_\alpha) = \int_{\mathbb{R}^{3n}\oplus L^2(\mathbb{R}^3,\mathbb{C}^2)} e^{2\pi i \Re\langle u_\alpha, z\rangle} \mathrm{d}\mu(z)$$

- Semiclassical convergence:

$$\gamma_\hbar \underset{\hbar\to 0}{\longrightarrow} \mathrm{d}\mu(u) \iff \lim_{\hbar\to 0} \hat{\gamma}_\hbar(^\forall u_\alpha) = \hat{\mu}(u_\alpha)$$

# Quantization

- Classical symbol:

$$F(u_\alpha) = f_{01}(\mathbf{q}, \mathbf{p}) + f_{02}(\alpha_1, \alpha_2) + f_{\mathrm{i}}(u_\alpha)$$

with the constraint that $f_{02}, f_{\mathrm{i}}$ are *polynomial* in $\alpha_1$ and $\alpha_2$ (the classical Hamiltonian for example).

- Quantization:

$$\hat{F}_\hbar = \mathrm{Op}_\hbar^{(\cdot)}(f_{01}) + \mathrm{Op}_\hbar^{\mathrm{Wick}}(f_{02}) + \mathrm{Op}_\hbar^{(\cdot),\mathrm{Wick}}(f_{\mathrm{i}})$$

where $\mathrm{Op}_\hbar^{\mathrm{Wick}}$ means that we substitute $\alpha_\lambda$ with $\hat{a}_\lambda$, $\bar{\alpha}_\lambda$ with $\hat{a}^*_\lambda$, and order all the $\hat{a}^*_\lambda$ on the left of the $\hat{a}_\lambda$.

- Semiclassical Limit:

$$\gamma_\hbar \underset{\hbar\to 0}{\longrightarrow} \mathrm{d}\mu(u) \implies \lim_{\hbar\to 0} \mathrm{Tr}\left(\gamma_\hbar \hat{F}_\hbar\right) = \int_{\mathbb{R}^{3n}\oplus L^2(\mathbb{R}^3,\mathbb{C}^2)} F(z)\mathrm{d}\mu(z) \ .$$

# The Correspondence Principle[2]

## Theorem 2 (Z. Ammari, MF, F. Hiroshima 2022)

$$\begin{array}{ccc} \gamma_\hbar & \xmapsto{\; e^{-\frac{i}{\hbar}t\hat{H}_\hbar}\,(\,\cdot\,)\,e^{\frac{i}{\hbar}t\hat{H}_\hbar} \;} & \gamma_\hbar(t) \\ \Big\downarrow {\scriptstyle \hbar\to 0} & & \Big\downarrow {\scriptstyle \hbar\to 0} \\ \mu_0 & \xmapsto[\; \mathscr{L}_t^{(\mathrm{N-M})}(\,\cdot\,) \;]{} & \mu_t \end{array}$$

[2] For coherent states and smooth charge distributions, Bohr's correspondence principle was established by A. Knowles, *PhD Thesis*, ETH Zürich, 2009.

## Remarks

- $t \mapsto \mu_t$ is dictated by the *Liouville transport equation* associated to the Newton-Maxwell system: $\mu_t = \mathscr{L}_t^{(\mathrm{N-M})}(\mu_0)$.
- The Newton-Maxwell Liouville flow "solves", as usual, the Newton-Maxwell equation in a much weaker form that the one we seek, stated in Theorem 1.
- The theorem above is an *Egorov-type theorem*, however it is weaker than the usual Egorov theorem due to the fact that this system has *infinitely many degrees of freedom*.

# Q-Driven Classical Trajectories – Part 2

# Proof of Theorem 1

- *A priori* uniqueness:

$$\varrho_{\forall j} \in \dot{H}^{-1} \cap \dot{H}^{1-\sigma} \implies \text{There exists at most one } H^{\sigma}\text{-solution of (N-M)}$$

- *Liouville flow:* [3]

$$[\textit{A priori}\ !\ ] \wedge [\ {}^{\exists}\mu_t = \mathscr{L}_t^{(\mathrm{N-M})}(\mu_0)] \implies {}^{\exists !}u_t \text{ sol. of (N-M) for } \mu_0\text{-a.a. } u_0$$

$^{\exists}\mu_t$ solution of N-M Liouville equation is yielded by Theorem 2

- *Saturating classical configurations via coherent states:*

$${}^{\forall}u_0\ {}^{\exists}\gamma_{\hbar}[u_0] \text{ (coherent state of minimal uncertainty): } \gamma_{\hbar}[u_0] \underset{\hbar\to 0}{\longrightarrow} \mathrm{d}\delta_{u_0}(u)$$

[3] C. Rouffort, *arXiv* 1809.01450, 2018.

# Outlook

# Future Developments

- Application to other models : There are other models, perhaps less interesting physically (Fröhlich polaron), where the quantum-to-classical features appear even more transparently (classical instability vs. quantum stability, quantum-driven classical dynamics,...)

- Diamagnetic Inequality : classical $E_0(0) = E_0(\mathbf{A})$; quantum $E_\hbar(0) < E_\hbar(\mathbf{A})$

- Charges with small radii : Moniz-Sharp on solid mathematical grounds

- Point Charges : Solve the quantum obstructions to point particles, and define the classical point dynamics by taking the "quantum detour"

# Thanks for the attention