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Classical and mean field limit of field-particle systems

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Outline

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Overview

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that should reduce classically to Newton equations

$$\begin{cases} \frac{d\xi}{dt}(t) = \frac{1}{m}\pi(t) \\ \frac{d\pi}{dt}(t) = -\nabla V(\xi(t)) \end{cases} \; .$$

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- Formally, we are looking to reduce a very big phase-space (the n particle one), to a single-particle phase space; in the limit $n \to \infty$.

Mean field theory for many bosons

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Consider the system of n non-relativistic bosons described by the following Hamiltonian of $L^2(\mathbb{R}^{nd})$:

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We expect that when n is very large the dynamics of each particle should be dictated by the mean field Hartree equation:

$$i\partial_t \varphi_t = -\Delta \varphi_t + (V * |\varphi_t|^2)\varphi_t \ .$$

Particles interacting with fields

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$$H = \frac{1}{2M}\int (\nabla\psi)^*(x)\nabla\psi(x)dx + \int \omega(k)a^*(k)a(k)dk + \lambda\int \varphi(x)\psi^*(x)\psi(x)dx$$

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with $\omega(k) = \sqrt{k^2+\mu^2}$, $M > 0$, $\mu \geq 0$, coupling constant $\lambda > 0$ and

$$\varphi(x) = \int \frac{\chi(k)}{\sqrt{\omega}}\Big(a(k)e^{ikx} + a^*(k)e^{-ikx}\Big)dk \; .$$

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(The precise meaning of the mean field limit in this system will be explained in detail.)

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where $p = -i\sqrt{\hbar}\nabla$, $q = \sqrt{\hbar}x$, $\omega(k) = c|k|$ and

$$A(x) = \sum_{\lambda=1,2} \int \sqrt{\frac{\hbar}{\omega(k)}} e_\lambda(k)\chi(k)\big(a(k,\lambda)e^{ik\cdot x} + a^*(k,\lambda)e^{-ik\cdot x}\big) dk \ .$$

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$$\begin{cases} \partial_t B + \nabla \times E = 0 \\ \partial_t E - \nabla \times B = -j \end{cases} \quad \begin{cases} \nabla \cdot E = \rho \\ \nabla \cdot B = 0 \end{cases}$$

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$$j = ev\varphi(\xi - x) \; , \quad \rho = e\varphi(\xi - x) \; .$$

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PROs: Applicability in a large number of systems (Hartree, Gross-Pitaevskii, Hartree-Fock, . . .).

CONs: Very specific initial states has to be considered (factorized states). No information on rate of convergence (apart for small times, in some systems); nor on fluctuations around the mean field solution.

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PROs: Informations on the evolution of fluctuations (between coherent states). Bound on the rate of convergence of reduced density matrices.

CONs: Specific initial states has to be considered, namely factorized and coherent ones (also partially factorized and linear combinations of the above [F., 1305]). Even if, due to symmetries, the natural setting of the system is $L^2(\mathbb{R}^{nd})$, the whole Fock space $\mathscr{F}(L^2(\mathbb{R}^d))$ has to be considered to prove convergence (since the method relies on the Weyl operators of the whole Fock space).

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CONs: No information on the fluctuations, nor on the rate of convergence.

- **Other approaches.**

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  Using the construction of a truncated Fock space (and of a map that substitutes the Weyl operators) [see Lewin et al., 2012] , they restrict the analysis to a space isomorphic to $L^2(\mathbb{R}^{nd})$, instead of considering the whole Fock space as in the Hepp method.

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  Also, I mention a work of Fröhlich et al. [2007], where they provide asymptotics for observables in the mean field and classical limit.

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- **Classical limit of Particle QED.** No result yet!

The Nelson model

The mean field limit as $\lambda \to 0$

Recall the Nelson Hamiltonian:

$$H = \frac{1}{2M} \int (\nabla \psi)^*(x) \nabla \psi(x) dx + \int \omega(k) a^*(k) a(k) dk + \lambda \int \varphi(x) \psi^*(x) \psi(x) dx \ .$$

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Consider a state ϕ_{n_1,n_2} such that $\langle \phi_{n_1,n_2} , (N_1 + N_2)\phi_{n_1,n_2} \rangle \sim n_1 + n_2$; we would like to describe its dynamics in the limit $n_1, n_2 \to \infty$ as a mean field theory, with the particles coupled as described above.

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In order to do that n_1 and n_2 has to be related, and it turns out that they have also to be related to the coupling constant λ, as $n_1 \sim n_2 \sim \lambda^{-2}$. So the mean field limit is also a weak coupling limit $\lambda \to 0$.

Quantum dynamics

Proposition

H is essentially self-adjoint on $\mathscr{F}_s(L^2(\mathbb{R}^3)) \otimes \mathscr{F}_s(L^2(\mathbb{R}^3))$.

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Consider the subspace $\mathscr{H}_{n_1} = L^2(\mathbb{R}^{3n_1}) \otimes \mathscr{F}_s(L^2(\mathbb{R}^3))$ of the whole Fock space with fixed number n_1 of non-relativistic particles, $H\Big|_{n_1}$ the restriction of H to that subspace.

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$$\|H_i\big|_{n_1}\phi\|^2 \leq \varepsilon^2 \|H_0\big|_{n_1}\phi\|^2 + C(\varepsilon, n_1)\|\phi\|^2 \ .$$

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→ Since ε does not depend on n_1, we can define H as the direct sum:

$$H = \bigoplus_{n_1=0}^{\infty} H\big|_{n_1} \,.$$

Classical dynamics

Recall the classical equation,

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A can be written as

$$A(x) = \int \frac{1}{\sqrt{\omega}}\left(\alpha(k)e^{ikx} + \bar{\alpha}(k)e^{-ikx}\right)dk \ .$$

The system of equations for u and α then becomes:

$$\begin{cases} i\partial_t u = -\frac{1}{2M}\Delta u + (2\pi)^{-3/2}(\check{\chi} * A)u \\ i\partial_t \alpha = \omega\alpha + (\omega)^{-1/2}\chi\big(\widehat{|u|^2}\big) \end{cases} .$$

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Proposition

Let (u_0, α_0) in $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. Then the equation above admits an unique global solution $(u(t), \alpha(t)) \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3))$.

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($\psi(u) = \int \bar{u}(x)\psi(x)dx$, $\psi^*(u) = (\psi(u))^*$, analogous for $a^{\#}(\alpha)$).

Then we can define the quantum evolution between coherent states as:

$$W(t,s) = C^*(u(t)/\lambda, \alpha(t)/\lambda) \exp\{-i(t-s)H\} C(u(s)/\lambda, \alpha(s)/\lambda) e^{i\Lambda(t,s)} \;;$$

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$$\Lambda(t,s) = -\frac{1}{2}(2\pi)^{-3/2}\lambda^{-2} \int_s^t \int_{\mathbb{R}^3} (\check{\chi} * A)(\tau)\bar{u}(\tau)u(\tau) dx d\tau \;.$$

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$U_2(t,s)$ can be rigorously defined by means of a truncated Dyson series *in the interaction representation*.

Theorem 1 ([F., 1301])

Let $\phi \in \mathscr{H}$. *Then*

$$\lim_{\lambda \to 0} W(t,s)\phi = U_2(t,s)\phi$$

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in the strong topology of $\mathscr{H}$; uniformly in t and s on compact intervals.

$U_2(t,s)$ describes the evolution of quantum fluctuations operators $\psi^{\#} - u(s)/\lambda$ and $a^{\#} - \alpha(s)/\lambda$ in the limit $\lambda \to 0$.

Mean field limit of quantum variables

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- We may consider the mean field limit for initial states that are coherent ($C(u_0/\lambda, \alpha_0/\lambda)\Omega$) or factorized ($u_0^{\otimes n_1} \otimes \alpha_0^{\otimes n_2}$) or factorized in the first species and coherent in the second one ($u_0^{\otimes n_1} \otimes C(\alpha_0/\lambda)\Omega_2$).
- Each one of these states has the property that $\langle \psi, (N_1 + N_2)\psi \rangle \sim n_1 + n_2 \sim \lambda^{-2}$.

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Theorem 2

As functions of $L^2(\mathbb{R}^3)$, we have the following convergence:

$$\lim_{\lambda\to 0}\langle C(u_0/\lambda, \alpha_0/\lambda)\Omega, \lambda\psi_t^{\#}(x)C(u_0/\lambda, \alpha_0/\lambda)\Omega\rangle = u^{\#}(t,x)\ ;$$

$$\lim_{\lambda\to 0}\langle C(u_0/\lambda, \alpha_0/\lambda)\Omega, \lambda a_t^{\#}(k)C(u_0/\lambda, \alpha_0/\lambda)\Omega\rangle = \alpha^{\#}(t,k)\ .$$

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- Up to a normalization factor, the result above (for products with the same number of creation and annihilation operators of each type) is equivalent to the convergence of reduced density matrices in the Hilbert-Schmidt norm.

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Theorem 3

As functions of $L^2(\mathbb{R}^6)$ and $L^2(\mathbb{R}^3)$ respectively, we have the following convergence:

$$\lim_{\lambda\to 0} \langle u_0^{\bigotimes n_1(\lambda)} \otimes \alpha_0^{\bigotimes n_2(\lambda)}, \lambda^2 \psi_t^*(x)\psi_t(y) u_0^{\bigotimes n_1(\lambda)} \otimes \alpha_0^{\bigotimes n_2(\lambda)} \rangle = \int_0^{2\pi} \bar{u}_\theta(t,x) u_\theta(t,y) \frac{d\theta}{2\pi}$$

$$\lim_{\lambda\to 0} \langle u_0^{\bigotimes n_1(\lambda)} \otimes \alpha_0^{\bigotimes n_2(\lambda)}, \lambda a_t^{\#}(k) u_0^{\bigotimes n_1(\lambda)} \otimes \alpha_0^{\bigotimes n_2(\lambda)} \rangle = \int_0^{2\pi} \alpha_\theta^{\#}(t,k) \frac{d\theta}{2\pi} .$$

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- Up to a normalization factor, the result above (for products with the same number of creation and annihilation operators of each type) is equivalent to the convergence of reduced density matrices in the Hilbert-Schmidt norm.
- The quantum evolution does not preserve the number n_2 of relativistic particles; this affects the classical limit, for initial states that have a fixed number of relativistic particles, in an unexpected fashion.

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We say that a family of density matrices $(\rho_\lambda)_{\lambda\in(0,\bar{\lambda})}$ on a Fock space $\mathscr{F}_s(\mathscr{Z})$ converges to a measure μ on $\mathscr{Z}$ if a probability measure on $\mathscr{Z}$ exists such that for all $\xi \in \mathscr{Z}$:

$$\lim_{\lambda\to 0} \mathrm{Tr}\Big[\rho_\lambda \exp\big\{i\big(a^*(\xi)+a(\xi)\big)/\sqrt{2}\big\}\Big] = \int_{\mathscr{Z}} e^{i\sqrt{2}\mathrm{Re}\langle\xi,\zeta\rangle_{\mathscr{Z}}} d\mu(\zeta)\ .$$

Proposition

Let $(\rho_\lambda)_{\lambda \in (0,\bar{\lambda})}$ *be a family of normal states on* $\mathscr{F}_s(L^2(\mathbb{R}^3)) \otimes \mathscr{F}_s(L^2(\mathbb{R}^3))$ *(satisfying some regularity properties) that converges to a probability measure* μ_0 *of* $\mathscr{Z} := L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ *when* $\lambda \to 0$*.*

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$$\lim_{\lambda\to 0}\gamma_\lambda^{p_1,p_2}(t) = \frac{1}{\int_{\mathscr{Z}}|z_1|^{2p_1}|z_2|^{2p_2}d\mu_t(z)}\int_{\mathscr{Z}}|z_1^{\otimes p_1}\otimes z_2^{\otimes p_2}\rangle\langle z_1^{\otimes p_1}\otimes z_2^{\otimes p_2}|d\mu_t(z)\ .$$

Future developments

- Classical limit of particle QED.

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- Mean field limit of the Nelson model without cut off.

- Classical limit of particle QED.
- Mean field limit of the Nelson model without cut off.
- Scattering in the mean field limit.

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Thank you.