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Cylindrical Wigner Measures in Bosonic QFTs

Grindelwald, September 13th, 2017.

Introduction. Semiclassical analysis meets constructive QFT

- Semiclassical analysis and pseudodifferential calculus are very powerful tools for the study of quantum mechanical non-relativistic systems.
- Relativistic quantum (field) theories should reproduce the "right" classical theory, in the limit $\hbar \to 0$. This is not *a priori* obvious, and it becomes a relevant question in some cases – *e.g.* in LQG [see Thiemann and Winkler, 2001, and references therein].
- Some of the challenges that make difficult to develop semiclassical analysis and pseudodifferential calculus in QFT are the following:
  - Existence of inequivalent irreducible representations of the algebra of canonical commutation relations;
  - Impossibility of defining a locally finite and translation invariant measure in infinite dimensional vector spaces;
  - Existence of one "privileged", representation dependent, quantization (Wick ordering);
  - Necessity of renormalization for interacting theories.

- Nonetheless, semiclassical analysis for infinite dimensional systems have been developed, with different approaches, and applied mostly to non-relativistic systems of many-bosons:
  - Krée and Raczka [1978]
  - Fröhlich, Graffi, Knowles and Schwarz [2007; 2009]
  - Ammari and Nier [2008; 2009; 2011; 2015]; Ammari and F. [2014; 2016]
  - Amour, Jager, R. Lascar and Nourrigat [2015a; 2015b; 2016a; 2016b; 2016c; 2017]
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    - F. [2016; 2017] – Framework for the semiclassical analysis of axiomatic QFT

Weyl C*-algebra

- $(X, \sigma) \in \mathbf{Symp}_{\mathbb{R}}$
- $\mathcal{W}_\hbar(X, \sigma) = \{W_\hbar(x), x \in X\}$:
  - $\forall x \in X\ ,\ W_\hbar(x) \neq 0$;
  - $\forall x \in X\ ,\ W_\hbar(x)^* = W_\hbar(-x)$;
  - $\forall x, y \in X\ ,\ W_\hbar(x) W_\hbar(y) = e^{-i\hbar\sigma(x,y)} W_\hbar(x+y)$.
- $\mathbb{W}_\hbar(X, \sigma) = \mathrm{C}^*\big(\mathcal{W}(X, \sigma)\big)$ Weyl C*-algebra.
- Fixed $(X, \sigma) \in \mathbf{Symp}_{\mathbb{R}}$, the Weyl C*-algebra $\mathbb{W}_\hbar(X, \sigma)$ is unique (up to *-isomorphisms).

Quantum states

- $\omega_\hbar \in \mathbb{W}_\hbar(X,\sigma)'$ is a quantum state;
- $\omega_\hbar \in \mathbb{W}_\hbar(X,\sigma)'_+$ is regular iff all $\mathbb{R}$-actions $\lambda \mapsto \omega_\hbar\big(W_\hbar(\lambda x)\big)$ are continuous:

$$\forall x \in X \; , \; \omega_\hbar\big(W_\hbar(\,\cdot\, x)\big) \in C(\mathbb{R},\mathbb{C}) \; .$$

- $\omega_\hbar \in \mathbb{W}_\hbar(X,\sigma)'_+$ is uniformly bounded iff

$$\sup_{\hbar\in(0,1)} \|\omega_\hbar\|_{\mathbb{W}_\hbar(X,\sigma)'} < \infty \; .$$

- $\omega_\hbar \in \mathbb{W}_\hbar(X,\sigma)'$ is regular (uniformly bounded) iff all four positive states in which it is decomposed are regular (uniformly bounded).
- The Fock vacuum $\Omega_\hbar \in \mathbb{W}_\hbar(H_\mathbb{R}, \mathrm{Im}\langle\cdot,\cdot\rangle_H)'_+$, $H \in \mathbf{Hilb}_\mathbb{C}$, with generating functional $\Omega_\hbar\big(W_\hbar(x)\big) = e^{-\frac{\hbar}{2}\|x\|_H^2}$ is the prototypical example of a regular and uniformly bounded quantum state.

Classical states

- Let $(X, \sigma) \in \mathbf{Symp}_{\mathbb{R}}$ be the symplectic space of a given Weyl C*-algebra $\mathbb{W}_{\hbar}(X, \sigma)$.

- Then $M \in \mathcal{M}_{\mathrm{cyl}}(V)_{\mathbb{C}}$ is a classical state iff
  - $V \in \mathbf{TVS}_{\mathbb{R}}$;
  - $V' \cong X$ (isomorphism in $\mathbf{VS}_{\mathbb{R}}$).

  (These assumptions can be relaxed slightly)

- $\mathcal{M}_{\mathrm{cyl}}(V)_{\mathbb{C}}$ is the set of (complex) cylindrical measures. $M \in \mathcal{M}_{\mathrm{cyl}}(V)_{\mathbb{C}}$ iff there exist four positive (or standard) cylindrical measures $M_{+,\mathbb{R}}, M_{-,\mathbb{R}}, M_{+,\mathbb{C}}, M_{-,\mathbb{C}} \in \mathcal{M}_{\mathrm{cyl}}(V)$ such that

$$M = M_{+,\mathbb{R}} - M_{-,\mathbb{R}} + i\left(M_{+,\mathbb{C}} - M_{-,\mathbb{C}}\right).$$

- A cylindrical measure on $V \in \mathbf{TVS}_{\mathbb{R}}$ (with respect to its continuous dual V') is equivalently:
  - A projective family of finite Radon measures $(\mu_\Phi, p_{\Phi\Psi})_{\Phi\supset\Psi\in F(V)}$, indexed by the set $F(V)$ of subspaces of V of finite codimension that are weakly $\sigma(V,V')$-closed. For any $\Phi\supset\Psi\in F(V)$, μ_Φ is a measure on the finite dimensional space V/Φ, $p_{\Phi\Psi}: V/\Phi \leftarrow V/\Psi$ is the canonical projection, and $\mu_\Phi = p_{\Phi\Psi\,*}\,\mu_\Psi$.
  - A finitely additive measure on the algebra $\mathcal{C}(V,V')$ generated by cylinders in V', that is countably additive when restricted to each (σ-)algebra generated by finite dimensional subspaces of V'. A cylinder $C_{x_1,\dots,x_n;b_1,\dots,b_n}\in\mathcal{C}(V,V')$, $\{x_j\}_{j=1}^n\subset V'$ and $\{b_j\}_{j=1}^n\subset\mathscr{B}(\mathbb{R})$, is defined by

$$C_{x_1,\dots,x_n;b_1,\dots,b_n} = \Bigl\{ v\in V\,,\ (\forall 1\leq j\leq n)\ x_j(v)\in b_j \Bigr\}\,.$$

- $\mathcal{M}_{\mathrm{cyl}}(V) \hookleftarrow \mathcal{M}_{\mathrm{rad}}(V)$ strictly. The Gaussian measure M_{G} with Fourier transform $\hat{M}_{\mathrm{G}}(x) = e^{-\frac{1}{2}\|x\|_2^2}$ on $L^2(\mathbb{R})$ is the prototypical example of a cylindrical measure that is not a Radon measure.

Linking quantum to classical states in AQFT: Semiclassical Locally Covariant QFT and “convergence” of functors

Locally covariant QFT

- Locally covariant QFT is an axiomatic formulation that generalizes the algebraic approach of Haag-Kastler to possibly curved spacetimes.
- (Bounded) regions of spacetime, and the corresponding spacetime mappings are conveniently collected into a *small category* (usually denoted by **Loc**). The relation between regions of spacetime and local algebras of quantum observables takes then the form of a *functor* $\mathscr{F} : \mathbf{Loc} \to \mathbf{C^*alg}$.
- This functor is assumed to satisfy four axioms (to be described later in detail): *Covariance*, *Isotony*, *Einstein Causality*, and *Time-slice*. In addition, we also assume the existence of locally covariant quantum fields.

Semiclassical functors

- SFO. $\{\mathscr{F}_\hbar : \mathbf{A} \to \mathbf{C^*alg}\,,\, \hbar \in (0,1)\}$:
  - $\mathbf{A}$ small;
  - $\forall a \in \mathrm{Obj}(\mathbf{A})$, $\exists (X_a, \sigma_a) \in \mathbf{Symp}_{\mathbb{R}}$, $\forall \hbar \in (0,1)$:

$$\mathbb{W}_\hbar(X_a, \sigma_a) \overset{\mathrm{w}_a}{\hookrightarrow} \mathscr{F}_\hbar(a)\ .$$

  - (Existence of locally $\substack{\text{co}\\ \text{contra}}$variant fields). $\forall f : a \to b$ $(a, b \in \mathrm{Obj}(\mathbf{A}))$,

$$\exists\ s_f : (X_a, \sigma_a) \substack{\longrightarrow\\ \longleftarrow} (X_b, \sigma_b)\ \ \text{(morphism in } \mathbf{Symp}_{\mathbb{R}})$$

such that

$$\mathscr{F}_\hbar(f)\big(\mathrm{w}_{\substack{a\\b}}\mathbb{W}_\hbar(X_{\substack{a\\b}}, \sigma_{\substack{a\\b}})\big) = \mathrm{w}_{\substack{b\\a}} \circ \mathfrak{s}_{f,\hbar}\big(\mathbb{W}_\hbar(X_{\substack{a\\b}}, \sigma_{\substack{a\\b}})\big)$$

where $\mathfrak{s}_{f,\hbar}$ is the isometric *-homomorphism defined by the action on the generators $\mathcal{W}(X_{\substack{a\\b}}, \sigma_{\substack{a\\b}})$:

$$(\forall x \in X_{\substack{a\\b}})\ ,\ \mathfrak{s}_{f,\hbar} W_\hbar(x) = W_\hbar(s_f(x))\ .$$

- SFS. $\{\mathscr{E}_\hbar : \mathbf{A} \to \mathbf{Ban}_C \,,\, \hbar \in (0,1)\} = \{\mathscr{D} \circ \mathscr{F}_\hbar \,,\, \hbar \in (0,1)\}$.

- CLF. $\mathscr{E}_0 : \mathbf{A} \to \mathbf{CylM}_{\mathbb{C}}$ is a *classical limit* of the SFS $\{\mathscr{E}_\hbar, \hbar \in (0,1)\}$ iff it has the same variance as the latter, and
  - $\forall a \in \mathrm{Obj}(\mathbf{A})$, $\exists V_a \in \mathbf{TVS}_{\mathbb{R}}$ such that $\mathscr{E}_0(a) = \mathcal{M}_{\mathrm{cyl}}(V_a)$ are classical states with respect to (X_a, σ_a);
  - $\forall f : a \to b$ ($a, b \in \mathrm{Obj}(\mathbf{A})$), $\mathscr{E}_0(f) : V_a \overset{\longleftarrow}{\longrightarrow} V_b$ is continuous with respect to the $\sigma(V_a, X_a)$, $\sigma(V_b, X_b)$ topologies, and satisfies ${}^{\mathrm{t}}\mathscr{E}_0(f) = s_f$.

Proposition

$\mathscr{E}_0$ always exists and it is unique up to natural isomorphisms.

- Concrete realization. $V_a = X_a^*$, with the $\sigma(X_a^*, X_a)$ weak topology; and $\mathscr{E}_0(f) = {}^{\mathrm{t}}s_f$ (algebraic transpose).

Semiclassical convergence

Theorem 1 (F. [2016])

$$\mathscr{E}_\hbar \underset{\hbar\to 0}{\rightharpoonup} \mathscr{E}_0$$

- (Wigner measures). $\left(a \mapsto \omega_{a,\hbar}\right) \underset{\hbar\to 0}{\longrightarrow} \left(a \mapsto M_a\right)$ (for reg. and unif. bounded quantum states).
- $\mathscr{E}_\hbar(f)$ converge "pointwise" to $\mathscr{E}_0(f)$, *i.e.* when applied to (suitable) states.
- The convergence is *surjective*: every local classical state $a \mapsto M_a$ is the limit $\hbar \to 0$ of some local quantum state.

Semiclassical consequences of the LCQFT axioms

Covariance

(Cov) $\big(\forall \hbar \in (0,1)\big)$, $\mathscr{F}_\hbar$ covariant functor.

$$\Downarrow$$

$\mathscr{E}_\hbar, \mathscr{E}_0$ contravariant functors: local classical fields are relativistically covariant.

Isotony

(Iso) $$\forall f \in \mathrm{Morph}(\mathbf{A})\,,\; f : a \hookrightarrow b \;\Rightarrow\; \mathscr{F}_\hbar(f) : \mathscr{F}_\hbar(a) \hookrightarrow \mathscr{F}_\hbar(b)\;.$$

$$\Downarrow$$

It is possible to define $\mathscr{M}_{\mathrm{cyl}}\Bigl(\varprojlim_{a\in \mathrm{Obj}(\mathbf{A})} V_a\Bigr)_{\mathbb{C}}$, the global classical states.

Einstein causality

(Ein)

$$\text{There is a monoidal structure on } \mathbf{A};$$
$$\big(\forall \hbar \in (0,1)\big)\ ,\ \mathscr{F}_\hbar^{\otimes} : \mathbf{A}^{\otimes} \to \mathbf{C^*alg}^{\otimes_{\min}} \text{ is homomorphic;}$$
$$\big(\forall a, b \in \mathrm{Obj}(\mathbf{A})\big)\ ,\ \mathbb{W}_\hbar(X_a \oplus X_b, \sigma_a \oplus \sigma_b) \overset{\mathrm{w}_{ab}}{\hookrightarrow} \mathscr{F}_\hbar^{\otimes}(a \otimes b)\ .$$

$$\Downarrow$$

There is a monoidal structure on $\mathbf{CylM}_{\mathbb{C}}$ and $\mathscr{E}_0^{\otimes}$ homomorphic;

$$\mathscr{E}_\hbar^{\otimes_{\min}} \underset{\hbar \to 0}{\rightharpoonup} \mathscr{E}_0^{\otimes}\ ;$$

and the convergence preserves or improves statistical independence.

Time-slice

- $\Sigma \in \mathrm{Cau}_{\mathbf{A}}$ iff
  - $\Sigma \in \mathrm{Obj}(\mathbf{Set})$;
  - $\Sigma = \varprojlim_{\Sigma \subset a \in \mathrm{Obj}(\mathbf{A})} a = \bigcap_{\Sigma \subset a \in \mathrm{Obj}(\mathbf{A})} a \neq \varnothing$; $i_a : \Sigma \to a$ canonical projection.
- $\mathscr{F}_\hbar(\Sigma) := \varprojlim_{\Sigma \subset a \in \mathrm{Obj}(\mathbf{A})} \mathscr{F}_\hbar(a)$.

$$\text{(TS)} \quad \begin{array}{c} \bigl(\forall \Sigma \in \mathrm{Cau}_{\mathbf{A}}\bigr) : \forall a \in \mathrm{Obj}(\mathbf{A}) \,,\, a \supset \Sigma \,,\, \forall \hbar \in (0,1) \,,\, \mathscr{F}_\hbar^{(\Sigma)}(i_a) \text{ isomorphism;} \\ \exists (X_\Sigma, \sigma_\Sigma) \in \mathbf{Symp}_{\mathbb{R}} \;,\; \forall \hbar \in (0,1) \;,\; \mathbb{W}_\hbar(X_\Sigma, \sigma_\Sigma) \overset{\mathrm{w}_\Sigma}{\hookrightarrow} \mathscr{F}_\hbar(\Sigma) \;. \end{array}$$

$$\Downarrow$$

$\bigl(\forall \Sigma \in \mathrm{Cau}_{\mathbf{A}}\bigr)$, $\mathcal{M}_{\mathrm{cyl}}(V_\Sigma)_{\mathbb{C}}$ are the cl. states of "time-sliced" fields;

Egorov theorems (that however need to be proved case-by-case).

Perspectives

- Semiclassical properties of “special states”: Vacuum states, KMS states (flat spacetime); Hadamard states (curved spacetime).
- Projective/inductive Wick/Weyl/Anti-Wick pseudodifferential calculus in non-Fock representations.
- Egorov theorems and characterization of the classical dynamics for theories with non-perturbative renormalization.

References

Z. Ammari and M. Falconi. Wigner measures approach to the classical limit of the Nelson model: Convergence of dynamics and ground state energy. *J. Stat. Phys.*, 157(2):330–362, 2014. doi: 10.1007/s10955-014-1079-7. URL http://dx.doi.org/10.1007/s10955-014-1079-7.

Z. Ammari and M. Falconi. Bohr's correspondence principle in quantum field theory and classical renormalization scheme: the nelson model. *ArXiv e-prints*, 2016. URL http://arxiv.org/abs/1602.03212.

Z. Ammari and F. Nier. Mean field limit for bosons and infinite dimensional phase-space analysis. *Ann. Henri Poincaré*, 9(8):1503–1574, 2008. ISSN 1424-0637. doi: 10.1007/s00023-008-0393-5. URL http://dx.doi.org/10.1007/s00023-008-0393-5.

Z. Ammari and F. Nier. Mean field limit for bosons and propagation of Wigner measures. *J. Math. Phys.*, 50(4):042107, 16, 2009. ISSN 0022-2488. doi: 10.1063/1.3115046. URL http://dx.doi.org/10.1063/1.3115046.

Z. Ammari and F. Nier. Mean field propagation of Wigner measures and BBGKY hierarchies for general bosonic states. *J. Math. Pures Appl. (9)*, 95(6):585–626, 2011. ISSN 0021-7824. doi: 10.1016/j.matpur.2010.12.004. URL http://dx.doi.org/10.1016/j.matpur.2010.12.004.

Z. Ammari and F. Nier. Mean field propagation of infinite dimensional Wigner measures with a singular two-body interaction potential. *Annali della Scuola Normale Sup.*, XIV:155–220, 2015.

L. Amour and J. Nourrigat. Hamiltonian systems and semiclassical dynamics for interacting spins in qed. *ArXiv e-prints*, 2015. URL https://arxiv.org/abs/1512.08429.

L. Amour and J. Nourrigat. Semiclassical expansion of the ground state for a model of interacting spins in qed. *ArXiv e-prints*, 2016. URL https://arxiv.org/abs/1603.08439.

L. Amour, L. Jager, and J. Nourrigat. On bounded pseudodifferential operators in Wiener spaces. *J. Funct. Anal.*, 269(9):2747–2812, 2015. ISSN 0022-1236. doi: 10.1016/j.jfa.2015.08.004. URL http://dx.doi.org/10.1016/j.jfa.2015.08.004.

L. Amour, R. Lascar, and J. Nourrigat. Beals characterization of pseudodifferential operators in wiener spaces. *Applied Mathematics Research eXpress*, 2016a. doi: 10.1093/amrx/abw001. URL http://amrx.oxfordjournals.org/content/early/2016/02/07/amrx.abw001.abstract.

L. Amour, R. Lascar, and J. Nourrigat. Weyl calculus in wiener spaces and in qed. *ArXiv e-prints*, 2016b. URL https://arxiv.org/abs/1610.06379.

L. Amour, R. Lascar, and J. Nourrigat. Weyl calculus in QED I. The unitary group. *J. Math. Phys.*, 58(1):013501, 24, 2017. ISSN 0022-2488. doi: 10.1063/1.4973742. URL http://dx.doi.org/10.1063/1.4973742.

M. Falconi. Cylindrical wigner measures. *ArXiv e-prints*, 2016. URL http://arxiv.org/abs/1605.04778.

M. Falconi. Concentration of cylindrical wigner measures. *Commun. Contemp. Math.*, 1750055, 2017. URL https://doi.org/10.1142/S0219199717500559.

J. Fröhlich, S. Graffi, and S. Schwarz. Mean-field- and classical limit of many-body Schrödinger dynamics for bosons. *Comm. Math. Phys.*, 271(3):681–697, 2007. ISSN 0010-3616. doi: 10.1007/s00220-007-0207-5. URL http://dx.doi.org/10.1007/s00220-007-0207-5.

J. Fröhlich, A. Knowles, and S. Schwarz. On the mean-field limit of bosons with Coulomb two-body interaction. *Comm. Math. Phys.*, 288(3):1023–1059, 2009. ISSN 0010-3616. doi: 10.1007/s00220-009-0754-z. URL http://dx.doi.org/10.1007/s00220-009-0754-z.

P. Krée and R. Raczka. Kernels and symbols of operators in quantum field theory. *Ann. Inst. H. Poincaré Sect. A (N.S.)*, 28(1):41–73, 1978.

T. Thiemann and O. Winkler. Gauge field theory coherent states (GCS). III. Ehrenfest theorems. *Classical Quantum Gravity*, 18(21):4629–4681, 2001. ISSN 0264-9381. doi: 10.1088/0264-9381/18/21/315. URL http://dx.doi.org/10.1088/0264-9381/18/21/315.