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Schrödinger-Klein-Gordon system as the classical limit of a Quantum Field Theory dynamics.

(Joint work with Zied Ammari)

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Outline

S-KG system

The S-KG system in dimension d.

(S-KG)
$$\begin{cases} i\partial_t u = -\dfrac{\Delta}{2M}u + Vu + (\varphi * A)u \\ (\Box + m^2)A = -\varphi * |u|^2 \end{cases}$$

- V is an external potential for the non-relativistic particle.
- φ can be a (regularizing) function, or Dirac's delta distribution.
- $M > 0$; $m \geq 0$.

The Yukawa interaction. ($d = 3$, $V = 0$, $\varphi = \delta$, $M = 1/2$, and $m = 1$)

(S-KG[δ])
$$\begin{cases} i\partial_t u = -\Delta u + Au \\ (\Box + 1)A = -|u|^2 \end{cases}$$

$$u(t_0) = u_0 \quad , \quad A(t_0) = A_0 \quad , \quad \partial_t A(t_0) = A_1 \quad .$$

In the literature, global well-posedness of the above system has been extensively investigated (e.g. Fukuda and Tsutsumi [1975]; Baillon and Chadam [1978]; Bachelot [1984]; Ginibre and Velo [2002]; Colliander, Holmer and Tzirakis [2008]; Pecher [2012]).

Theorem (Pecher [2012])

Let $0 \leq s \leq \sigma \leq s+1$ *and* $u_0 \in H^s(\mathbb{R}^3)$, $A_0 \in H^\sigma(\mathbb{R}^3)$, $A_1 \in H^{\sigma-1}(\mathbb{R}^3)$.
Then (S-KG[δ]) *is globally well-posed; i.e. there exists a unique solution* $u \in \mathscr{C}^0(\mathbb{R}, H^s(\mathbb{R}^3))$, $A \in \mathscr{C}^0(\mathbb{R}, H^\sigma(\mathbb{R}^3)) \cap \mathscr{C}^1(\mathbb{R}, H^{\sigma-1}(\mathbb{R}^3))$.

The complex K-G field.

Let $\omega(k) = \sqrt{k^2 + m^2}$. In place of A, it is useful to utilize the complex field α defined by:

$$A(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\omega(k)}} \big(\bar{\alpha}(k) e^{-ik\cdot x} + \alpha(k) e^{ik\cdot x}\big)\, dk$$

Then, with $\chi = (2\pi)^{\frac{d}{2}} \hat{\varphi}$, (S-KG) is equivalent to a system of equations for u and α.

Global well-posedness of (S-KG[χ])

(S-KG[χ])
$$\begin{cases} i\partial_t u = -\dfrac{\Delta}{2M}u + Vu + (\varphi * A)u \\ i\partial_t \alpha = \omega\alpha + \dfrac{\chi}{\sqrt{2\omega}}\widehat{(\bar{u}u)} \end{cases}$$

Proposition

Let $V \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}_+)$, $\chi/\sqrt{\omega} \in L^2(\mathbb{R}^d)$; *and* $u_0 \in L^2(\mathbb{R}^d)$, $\alpha_0 \in L^2(\mathbb{R}^d)$.

Then (S-KG[χ]) *is globally well-posed; i.e. there exists a unique solution* $u \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^d))$, $\alpha \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^d))$.

The energy functional.

(S-KG[χ]) is an Hamiltonian system. Consider the following (densely defined) functional on $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d) \supset D(h) \to \mathbb{R}$:

$$h(u,\alpha) = \int_{\mathbb{R}^d} \bar{u}(x)\Big(-\frac{\Delta_x}{2M} + V(x)\Big)u(x)dx + \int_{\mathbb{R}^d} \bar{\alpha}(k)\omega(k)\alpha(k)dk$$
$$+\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^{2d}} \bar{u}(x)\frac{\chi(k)}{\sqrt{2\omega(k)}}\Big(\bar{\alpha}(k)e^{-ik\cdot x} + \alpha(k)e^{ik\cdot x}\Big)u(x)dxdk$$

Then (S-KG[χ]) can be rewritten as:

$$i\partial_t \begin{pmatrix} u \\ \alpha \end{pmatrix} = \begin{pmatrix} \frac{\delta h}{\delta \bar{u}} \\ \frac{\delta h}{\delta \bar{\alpha}} \end{pmatrix}$$

Nonlinear flow.

The global well-posedness results on (S-KG[χ]) implies that there exists a unique map (the Hamiltonian flow) $(\Phi_h)_{t_0}^t$, associated to h, such that for any $t_0, t \in \mathbb{R}$:

- $$(\Phi_h)_{t_0}^t : L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$$
- $$(\Phi_h)_{t_0}^t \begin{pmatrix} u_0 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} u_{t_0}(t) \\ \alpha_{t_0}(t) \end{pmatrix}$$

  where $\big(u_{t_0}(t), \alpha_{t_0}(t)\big)$ is the solution of the Cauchy problem associated with (S-KG[χ]) with initial datum $\big(u_{t_0}(t_0), \alpha_{t_0}(t_0)\big) = \big(u_0, \alpha_0\big)$.
- In this system the mass of the Schrödinger particle is conserved, i.e. $\|u_{t_0}(t)\|_2 = \|u_0\|_2$ for any $t \in \mathbb{R}$. Formally, also the energy $h(u \oplus \alpha)$ is conserved.

The model of bosonic QFT

The (symmetric) Fock space.

- Let $\mathscr{H}$ be a separable Hilbert space. For any $n \in \mathbb{N}^*$, we define the n-particle space to be

$$\mathscr{H}_n = \underbrace{\mathscr{H} \otimes_s \mathscr{H} \otimes_s \cdots \otimes_s \mathscr{H}}_{n}$$

- Also we define the vacuum space to be

$$\mathscr{H}_0 = \mathbb{C}$$

- The symmetric Fock space $\Gamma_s(\mathscr{H})$ is then defined as the direct sum:

$$\Gamma_s(\mathscr{H}) = \bigoplus_{n=0}^{\infty} \mathscr{H}_n$$

- A vector of $\Gamma_s(\mathscr{H})$ may be written as a collection $\psi = (\psi_0, \dots, \psi_n, \dots)$, with $\psi_n \in \mathscr{H}_n$ for any $n \in \mathbb{N}$.

- $\Gamma_s(\mathscr{H})$ is a Hilbert space with the scalar product:

$$\langle \psi, \phi \rangle_{\Gamma_s(\mathscr{H})} = \sum_{n=0}^{\infty} \langle \psi_n, \phi_n \rangle_{\mathscr{H}_n}$$

Annihilation and creation operators. [on $\Gamma_s(L^2(\mathbb{R}^d))$]

- The annihilation operator (valued distribution) $a(x)$ makes a vector "jump down" from a state with n particles, to one with $n-1$ particles (destroys a particle).

$$(a(x)\psi)_n(X_n) = \sqrt{n+1}\psi_{n+1}(x, X_n)$$

- Analogously, the creation operator (valued distribution) $a^*(x)$ creates a particle, making a vector "jump up" from n to $n+1$ particles.

$$(a^*(x)\psi)_n(X_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta(x - x_j)\psi_{n-1}(X_n \setminus x_j)$$

- They satisfy the Canonical Commutation Relations (CCR)

$$[a(x), a^*(y)] = \varepsilon\delta(x - y)$$

$\text{Classical}_{\varepsilon=0} \longrightarrow \text{Quantum}_{\varepsilon>0}$ Dictionary.

- $$L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d) \longrightarrow \Gamma_s(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$$
  Classical phase space (infinite dim.) $\longrightarrow$ Quantum Fock space
- $$u(x), \bar{u}(x) \text{ and } \alpha(k), \bar{\alpha}(k) \longrightarrow \psi(x), \psi^*(x) \text{ and } a(k), a^*(k)$$
  Classical variables (scalar fields) $\longrightarrow$ Quantum variables (op.valued distributions)
- $$f(u, \alpha) : D(f) \to \mathbb{R} \longrightarrow F := f(\psi, a) \in \mathscr{C}(\Gamma_s)\ ,\ F = F^*$$
  Classical observables (functionals) $\longrightarrow$ Quantum observables (s.-a. operators)
- $$(\Phi_h)^t_{t_0} \longrightarrow e^{-\frac{i}{\varepsilon}(t-t_0)H}\ ,\ H = h(\psi, a)$$
  Classical evolution (Ham. flow on phase sp.) $\longrightarrow$ Quantum evolution (unitary group on Fock sp.)

On the quantum level everything depends (implicitly or explicitly) on ε

The quantum Hamiltonian: Nelson model.

$$H = \int_{\mathbb{R}^d} \psi^*(x)\Big(-\frac{\Delta_x}{2M} + V(x)\Big)\psi(x)dx + \int_{\mathbb{R}^d} a^*(k)\omega(k)a(k)dk$$
$$+\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^{2d}} \psi^*(x)\frac{\chi(k)}{\sqrt{2\omega(k)}}\Big(a^*(k)e^{-ik\cdot x} + a(k)e^{ik\cdot x}\Big)\psi(x)dxdk$$

- For any $\omega^{-1/2}\chi \in L^2(\mathbb{R}^d)$, H is an unbounded self-adjoint operator on $\Gamma_s(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$.
- If $\chi = 1$, H is ill-defined. A renormalization procedure is necessary (Nelson [1964]).

The classical limit $\varepsilon \to 0$

Some bibliographic remarks.

The classical (mean field) limit for infinite dimensional systems has been widely studied. A (not comprehensive) reference list of the developed approaches:

- Coherent states (Hepp) method.
  - Hepp [1974]
  - Ginibre and Velo [1979, 1980]; Ginibre, Nironi and Velo [2006]; F. [2013]
  - Rodnianski and Schlein [2009]; Grillakis, Machedon and Margetis [2010]; Chen, Lee and Schlein [2011]
- Reduced density matrices (BBGKY hierarchy, direct counting)
  - Spohn [1980]
  - Bardos, Golse and Mauser [2000]; Bardos, Erdős, Golse, Mauser and Yau [2002]
  - Erdős and Yau [2001]; Erdős, Schlein and Yau [2007, 2010]
  - T. Chen and Pavlović [2011]; T. Chen, Hainzl, Pavlović and Seiringer [2013, 2014]; X. Chen and Holmer [2013]
  - Knowles and Pickl [2010]; Pickl [2011]
  - Lewin, Nam and Rougerie [2013]

- Dyson tree expansion
  - Fröhlich, Graffi and Schwarz [2007]; Fröhlich, Knowles and Schwarz [2009]
- Truncated Fock space
  - Lewin, Nam and Schlein [2013]
- Wigner measures
  - Ammari and Nier [2008, 2009, 2011]
  - Pawilowski and Liard [2014]
  - Ammari and F. [2014]

Quantum states.

- A quantum state of $\Gamma_s(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$ may be thought as a $\mathbb{C}$-linear map $R : \mathscr{L}(\Gamma_s) \to \mathbb{C}$ on the C^*-algebra $\mathscr{L}(\Gamma_s)$ that satisfies:
  - For any $X \in \mathscr{L}(\Gamma_s)$, $R(X^*X) \geq 0$.
  - Let $1 \in \mathscr{L}(\Gamma_s)$ be the identity operator, then $R(1) = 1$.
- Given a state R and a bounded observable X, $R(X)$ is the expectation value of X on the state R.
- R can be put in $1-1$ correspondence with positive trace class operators $\rho \in \mathscr{L}^1(\Gamma_s)$ with $\mathrm{Tr}[\rho] = 1$, by

$$R(\cdot) = \mathrm{Tr}[\rho\,\cdot\,]$$

- The action of a state ρ extends naturally to an unbounded observable H, provided $|\mathrm{Tr}[\rho H]| < +\infty$.
- In general a quantum state depends on ε. We will write ρ_ε as a reminder.

Evolution of a state.

- he unitary group $e^{-\frac{i}{\varepsilon}tH}$, with H the Nelson Hamiltonian, dictates the evolution of quantum states.
- $\rho_\varepsilon(t) = e^{-\frac{i}{\varepsilon}tH}\rho_\varepsilon e^{\frac{i}{\varepsilon}tH}$ (Schrödinger picture).
- By the cyclicity of Tr, the time evolution may be put on the observable $X(t) = e^{\frac{i}{\varepsilon}tH}Xe^{-\frac{i}{\varepsilon}tH}$ (Heisenberg picture):
$$\mathrm{Tr}[\rho_\varepsilon(t)X] = \mathrm{Tr}[\rho_\varepsilon X(t)]$$

$$? \underset{\varepsilon \to 0}{\longleftarrow} \mathrm{Tr}[\rho_\varepsilon \,\cdot\,].$$

- Quantum states were not part of the dictionary.
- The probabilistic interpretation of quantum systems suggests they should converge to some probabilistic object on the classical phase space.
  (probability distribution?)
- Sequences of states converge, in the classical limit, to Wigner measures.
- In finite dimensional phase spaces, Wigner measures have been extensively studied (e.g. Colin de Verdière [1985]; Helffer, Martinez and Robert [1987]; Tartar [1990]; Gérard [1991]; Lions and Paul [1993]; Nier [1996]; Gérard, Markowich, Mauser and Poupaud [1997]).
- The concept has been extended to infinite dimensional phase spaces by Ammari and Nier [2008, 2009, 2011].

Quantum states and Wigner measures $(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$.

- The Wigner measures μ are probability measures on $\mathscr{Z} := L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$, i.e. they satisfy $(z \in \mathscr{Z},\ z = z_1 \oplus z_2)$:

$$\mu(\mathscr{Z}) = \int_{\mathscr{Z}} d\mu(z) = 1$$

- Let $\mathscr{Z} \ni \xi = \xi_1 \oplus \xi_2$. Define the unitary Weyl operator $W(\xi)$ on $\Gamma_s(\mathscr{Z})$ as:

$$W(\xi) = e^{\frac{i}{\sqrt{2}} \int (\xi_1 \psi^* + \bar{\xi}_1 \psi) dx} e^{\frac{i}{\sqrt{2}} \int (\xi_2 a^* + \bar{\xi}_2 a) dk}$$

- Given a family of states $\big(\mathrm{Tr}[\rho_\varepsilon \cdot\,]\big)_{\varepsilon \in (0,\bar{\varepsilon})}$ we say that μ is associated to it if for any $\xi \in \mathscr{Z}$ there exists a subset $\mathscr{E} \subset (0, \bar{\varepsilon})$ (with $0 \in \overline{\mathscr{E}}$) such that:

$$\lim_{\varepsilon \to 0, \varepsilon \in \mathscr{E}} \mathrm{Tr}[\rho_\varepsilon W(\xi)] = \int_{\mathscr{Z}} e^{\sqrt{2} i (\mathrm{Re}\langle \xi_1, z_1 \rangle_2 + \mathrm{Re}\langle \xi_2, z_2 \rangle_2)} d\mu(z)$$

- Under suitable assumptions on the family $\big(\mathrm{Tr}[\rho_\varepsilon \cdot]\big)_{\varepsilon\in(0,\bar{\varepsilon})}$, it is always possible to extract a subsequence that converges to a unique Wigner measure μ.

- We shall consider, without loss of generality, only families of states with a single associated Wigner measure.

Time evolution.

- Are we able to say something about $\mu_{t_0}(t)$, the Wigner measure associated to $\big(\mathrm{Tr}[\rho_\varepsilon(t-t_0)\,\cdot\,]\big)_{\varepsilon\in(0,\bar{\varepsilon})}$? (reminder: $\rho_\varepsilon(t-t_0) = e^{-\frac{i}{\varepsilon}(t-t_0)H}\rho_\varepsilon e^{\frac{i}{\varepsilon}(t-t_0)H}$)

- The answer is affirmative:

$$\mu_{t_0}(t) = (\Phi_h)^t_{t_0}\,\#\,\mu_{t_0}$$

(μ_{t_0} is the initial measure, associated to $\big(\mathrm{Tr}[\rho_\varepsilon\,\cdot\,]\big)_{\varepsilon\in(0,\bar{\varepsilon})}$, and $(\Phi_h)^t_{t_0}$ is the classical flow)

- In other words, given a (suitable) self-adjoint observable F on $\Gamma_s(\mathscr{Z})$ that is the quantization of a real functional $f(z)$, $z \in \mathscr{Z}$:

$$\lim_{\varepsilon\to 0} \mathrm{Tr}[\rho_\varepsilon(t-t_0)F] = \int_{\mathscr{Z}} f(z) d\mu_{t_0}(t,z) = \int_{\mathscr{Z}} f(z_{t_0}(t)) d\mu_{t_0}(z_0)$$

where $z_{t_0}(t)$ is the solution of (S-KG[χ]) with initial datum $z_{t_0}(t_0) = z_0$.

Theorem (Ammari and F. [2014]) (reminder: $z = z_1 \oplus z_2 = u \oplus \alpha$)

- $$L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d) \underset{\varepsilon\to 0}{\longleftarrow} \Gamma_s(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$$
  Wigner measures μ — Quantum states $\mathrm{Tr}[\rho_\varepsilon \cdot]$
- $$\text{(S-KG}[\chi]\text{) flow } (\Phi_h)_{t_0}^t \underset{\varepsilon\to 0}{\longleftarrow} \text{Nelson dynamics } e^{-\frac{i}{\varepsilon}(t-t_0)H}$$
  Evolved Wigner m. $\mu_{t_0}(t) = (\Phi_h)_{t_0}^t \, \# \, \mu_{t_0}$ — Evolved state $\mathrm{Tr}[e^{-\frac{i}{\varepsilon}(t-t_0)H}\rho_\varepsilon e^{\frac{i}{\varepsilon}(t-t_0)H} \cdot]$
- $$u(x), \bar{u}(x) \text{ and } \alpha(k), \bar{\alpha}(k) \underset{\varepsilon\to 0}{\longleftarrow} \psi(x), \psi^*(x) \text{ and } a(k), a^*(k)$$
  (averaged) Orth. proj. $\int_{\mathscr{Z}} |z_1\rangle\langle z_1| d\mu_{t_0}(t,z)$ — 1-part. reduced density matrix $\gamma^{(1)}_{\varepsilon,\psi}(t-t_0)$
- $$\text{Cl. observable } f(u,\alpha) \underset{\varepsilon\to 0}{\longleftarrow} \text{Quant. observable } F = f(\psi, a)$$
  Average on $\mathscr{Z}$: $\int_{\mathscr{Z}} f(z_{t_0}(t)) d\mu_{t_0}(z_0)$ — Expectation of F: $\mathrm{Tr}[\rho_\varepsilon(t)F]$

Ideas of the proof.

- At the quantum level, justify the following integral (Duhamel) formula:

$$\mathrm{Tr}\Big[\rho_\varepsilon(t) W(\xi)\Big] = \mathrm{Tr}\Big[\rho_\varepsilon W(\xi)\Big] + \frac{i}{\varepsilon}\int_0^t \mathrm{Tr}\Big[\rho_\varepsilon(s)[H_I, W(\xi(s))]\Big] ds$$

- Taking the limit $\varepsilon \to 0$, obtain a weak transport equation that is satisfied by the probability-measure-valued map $t \in \mathbb{R} \to \mu(t)$:

$$\forall f \in \mathcal{C}^\infty_{0,cyl}(\mathbb{R} \times \mathscr{Z}), \quad \int_{\mathbb{R}}\int_{\mathscr{Z}} \left(\partial_t f + \mathrm{Re}\langle \nabla f, \mathscr{V}_t\rangle\right) \, d\mu_t dt = 0$$

where $\mathscr{V}_t$ is a velocity vector field. ($f : \mathscr{Z} \to \mathbb{R}$ is cylindrical if there is a finite rank orthogonal projection $\wp$ on $\mathscr{Z}$ and a function $g : \wp\mathscr{Z} \to \mathbb{R}$ such that $f(z) = g(\wp z)$ for any $z \in \mathscr{Z}$)

- $\mu(t) = (\Phi_h)_0^t{}_{\#}\,\mu(0)$ is a solution of the transport equation, but a proof of uniqueness is needed. This is done extending a method introduced in finite dimensions by Ambrosio, Gigli and Savaré [2005].

- Extend the result to a general class of states and observables by approximation arguments.

Final remarks

"Transferred" properties?

- The renormalization process of the Nelson model can be understood on the classical level (Ammari and F., work in progress).

- The dispersive properties of the classical system may lead to a better understanding of the long-time behaviour of the quantum system? Also, can we prove a result for scattering in the classical limit?

- On the quantum level, also in the singular situation $\chi = 1$, the dynamics can be defined with an external confining potential $V \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}_+)$. Is it an indication that (S-KG) may have a global solution in some suitable space even with $\varphi = \delta$ and $V \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}_+)$?

Thank you for the attention.

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